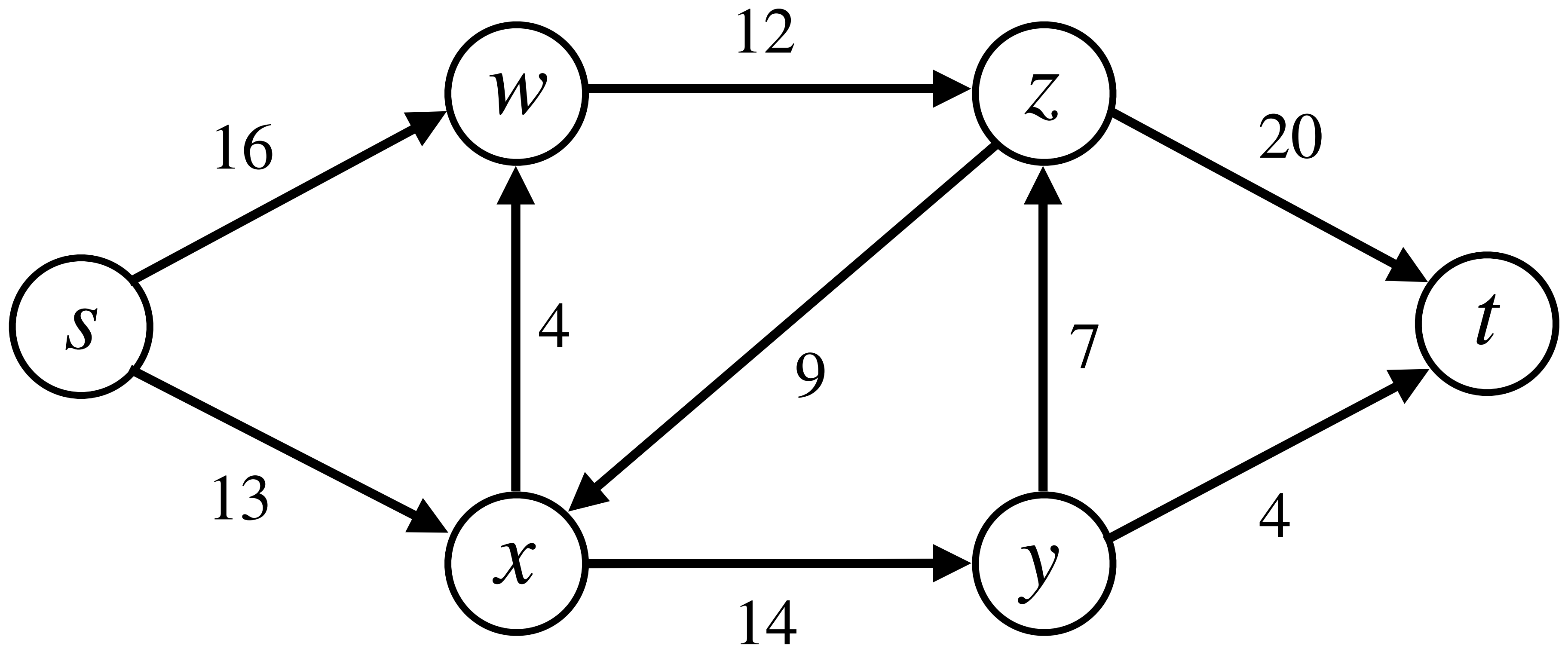
Lecture 17

Flow Networks, Ford-Fulkerson Method

Source: Introduction to Algorithms, CLRS and Kleinberg & Tardos

Flow Networks

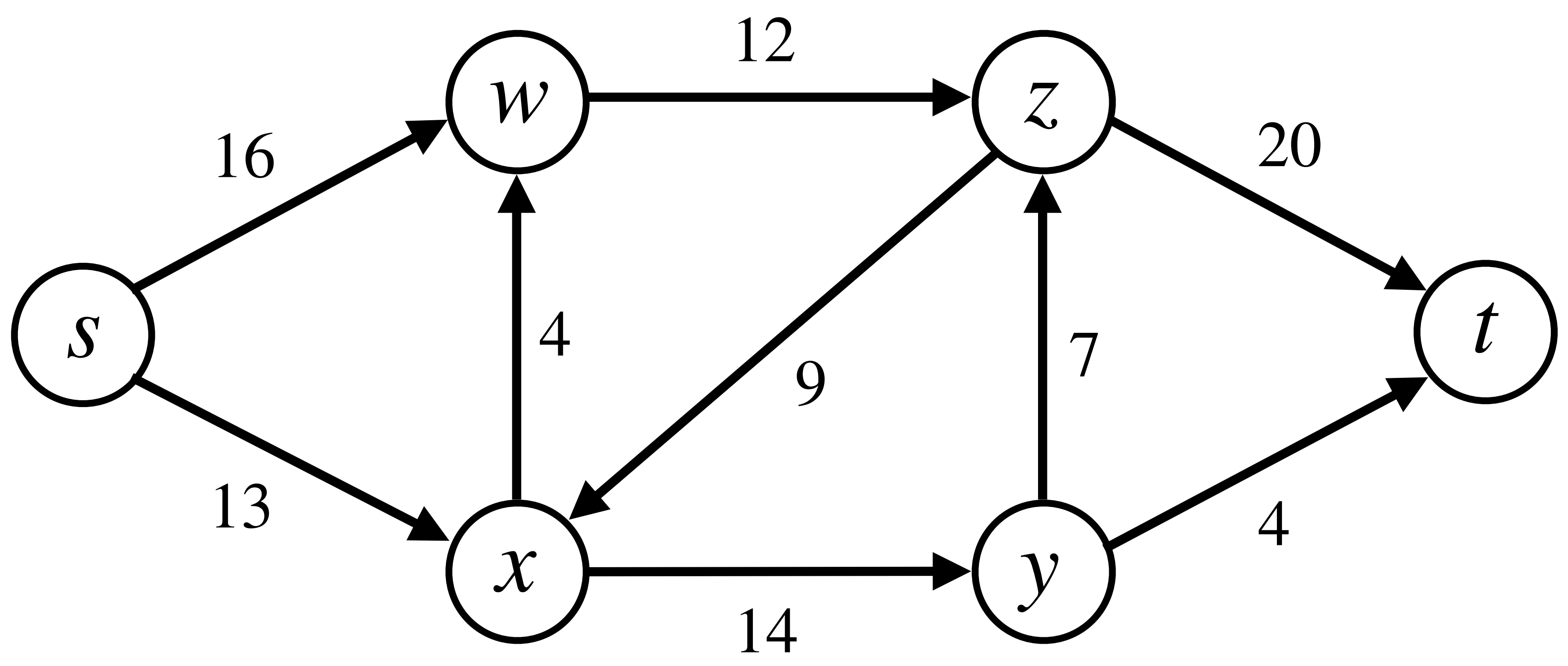
Flow Networks



(a)

Flow Networks

Figure (a) is flow network of a shipping company, where:



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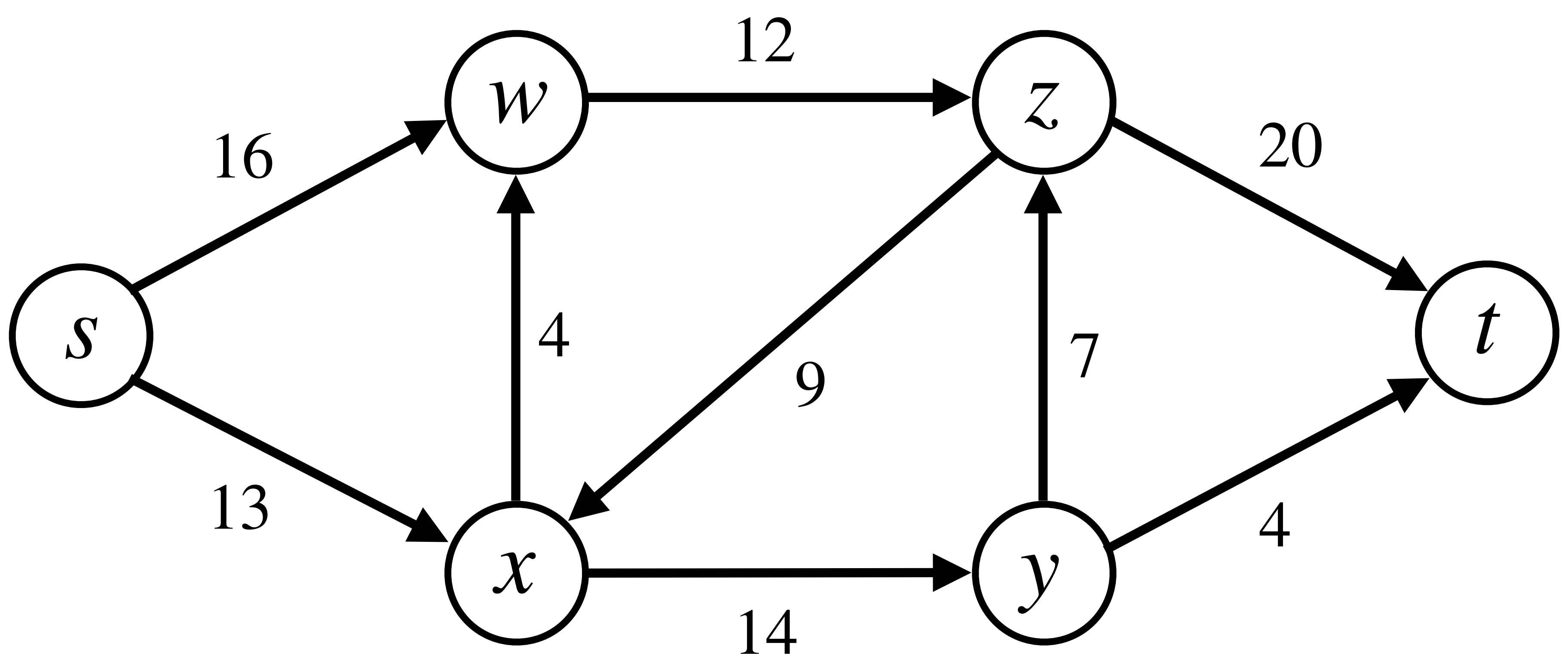

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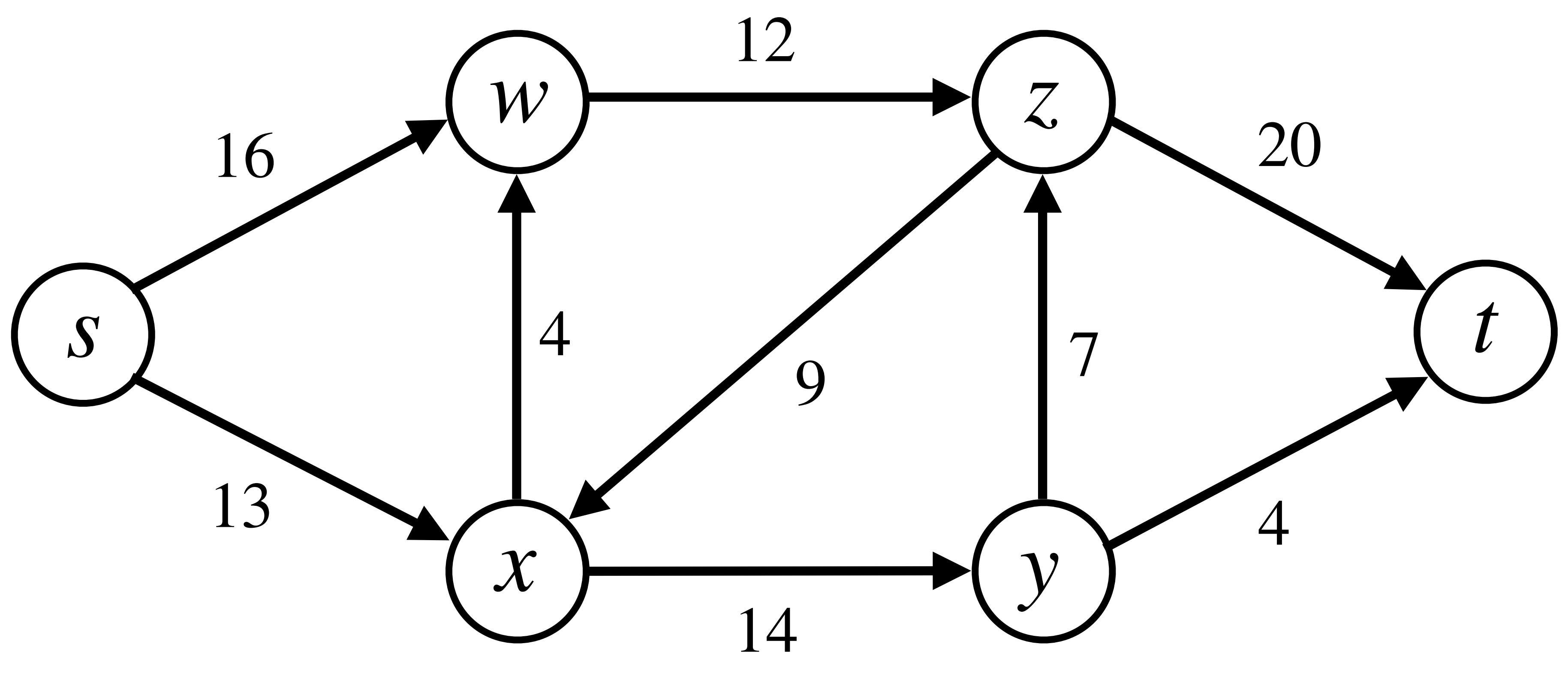
- Vertices represent **cities**. s & t are the **source** & **sink** cities.
- The number on any (u, v) edge is the **maximum number of packets** that can go from u to v per day.



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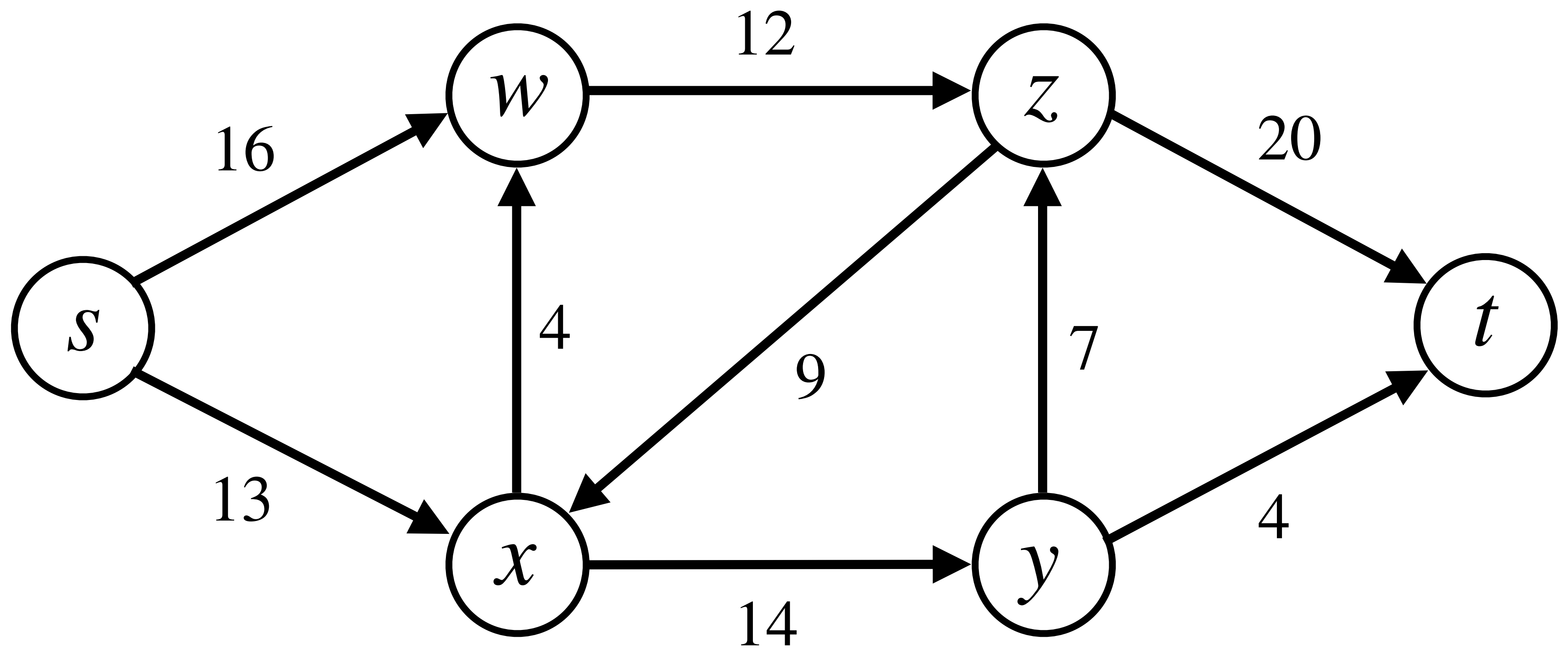

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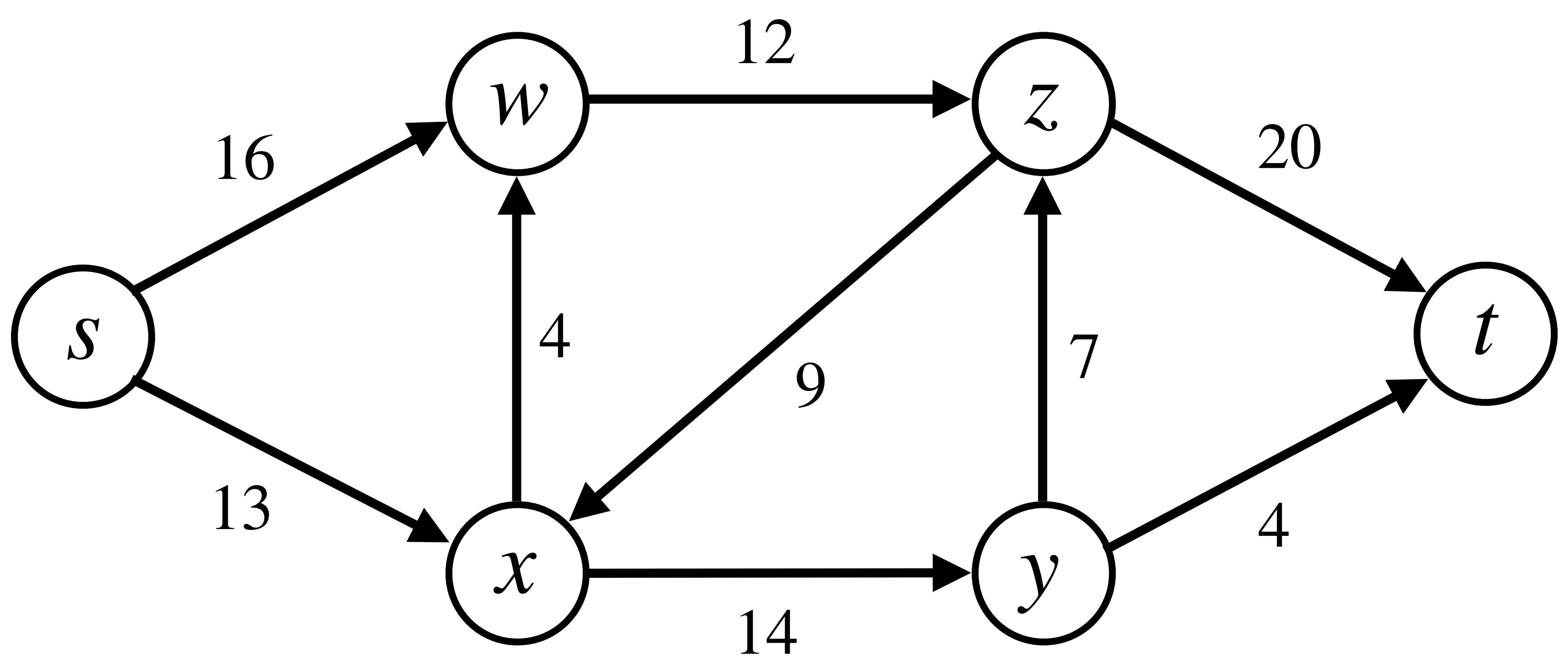
Goal: Find the **maximum number of packets** that can be shipped from s if the packets received and



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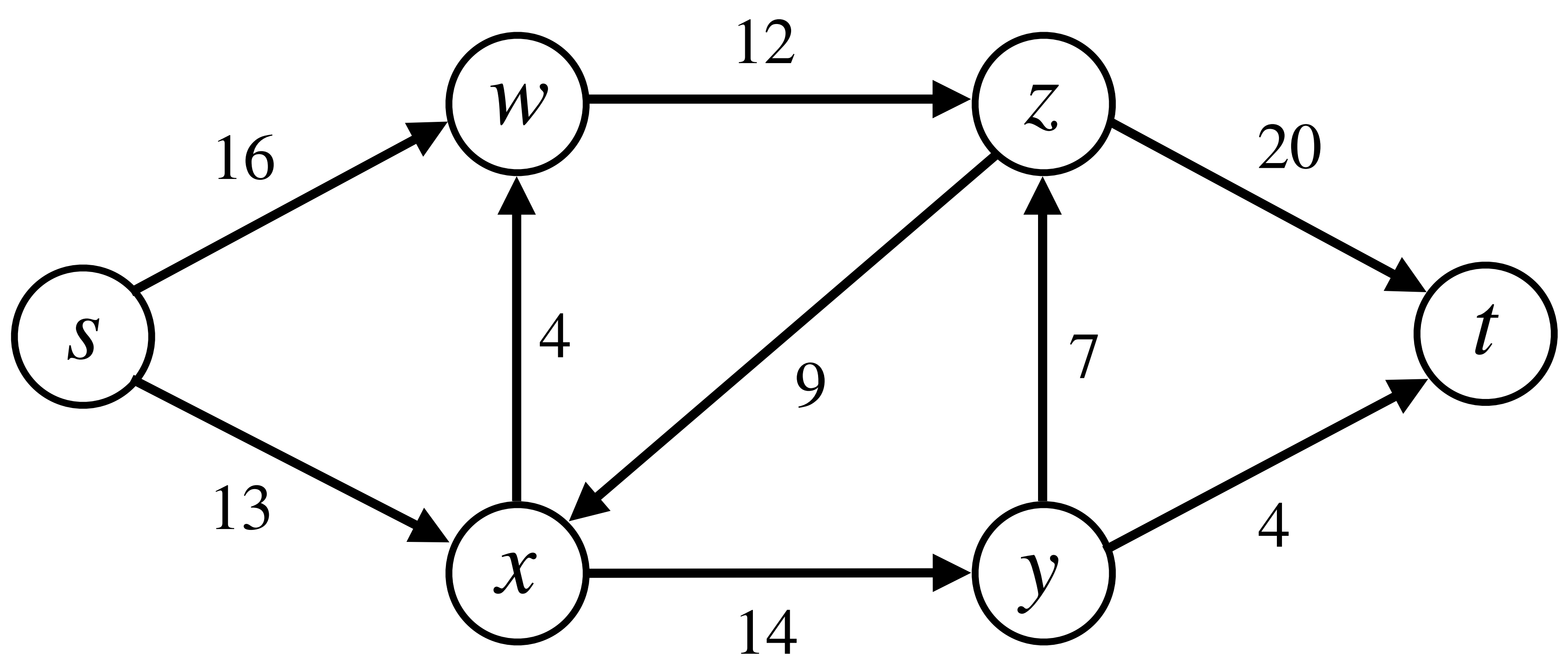

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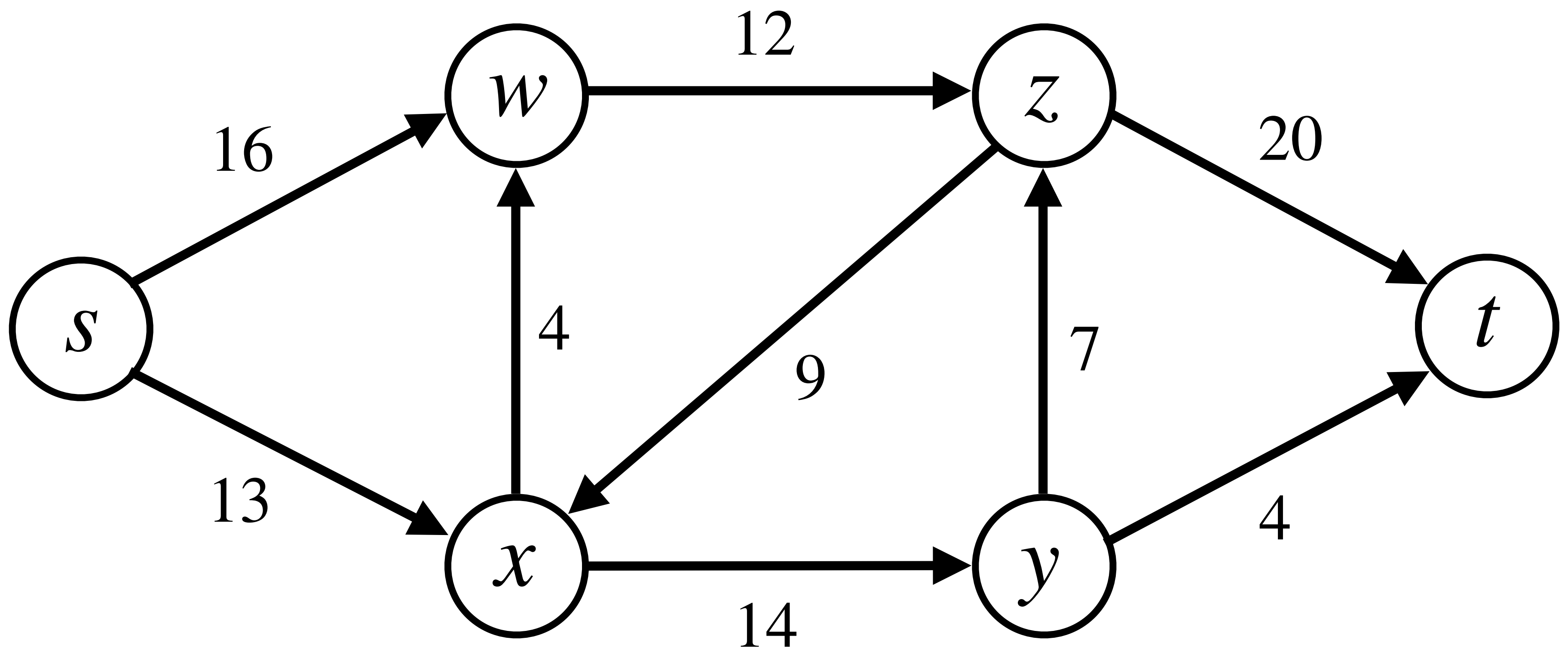

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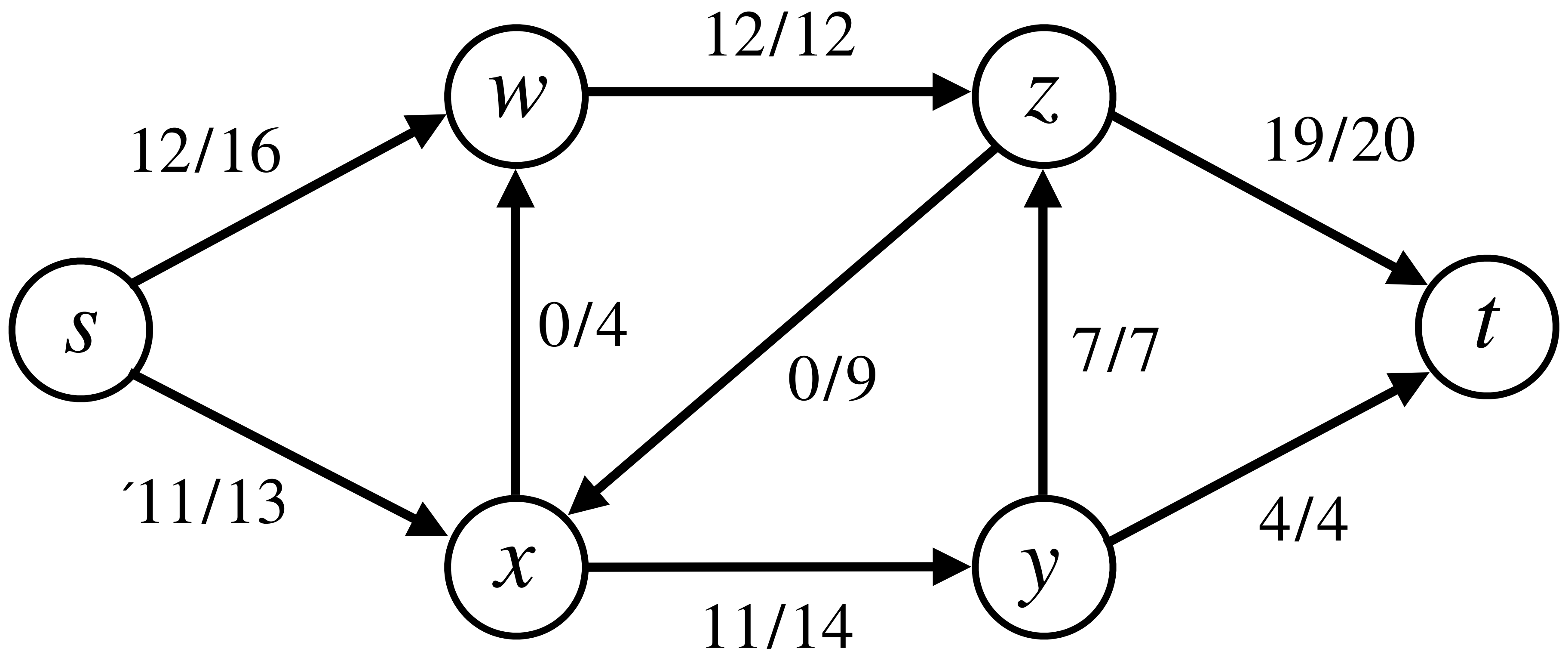

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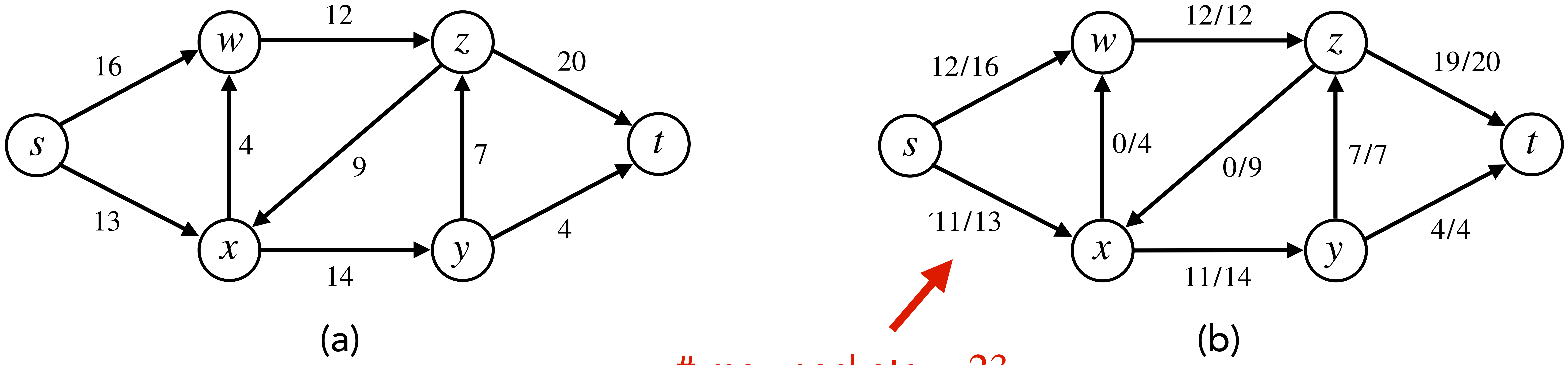
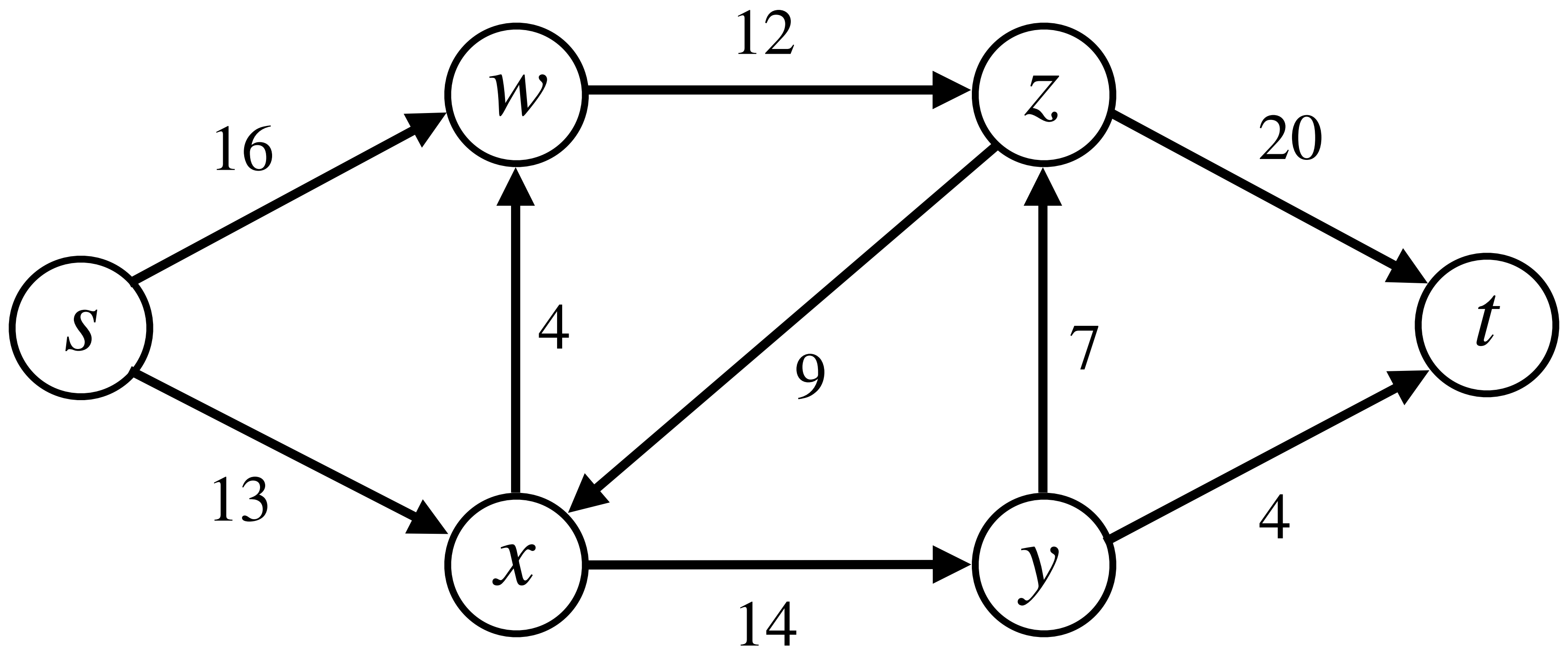


max packets $= 23$

Flow Networks



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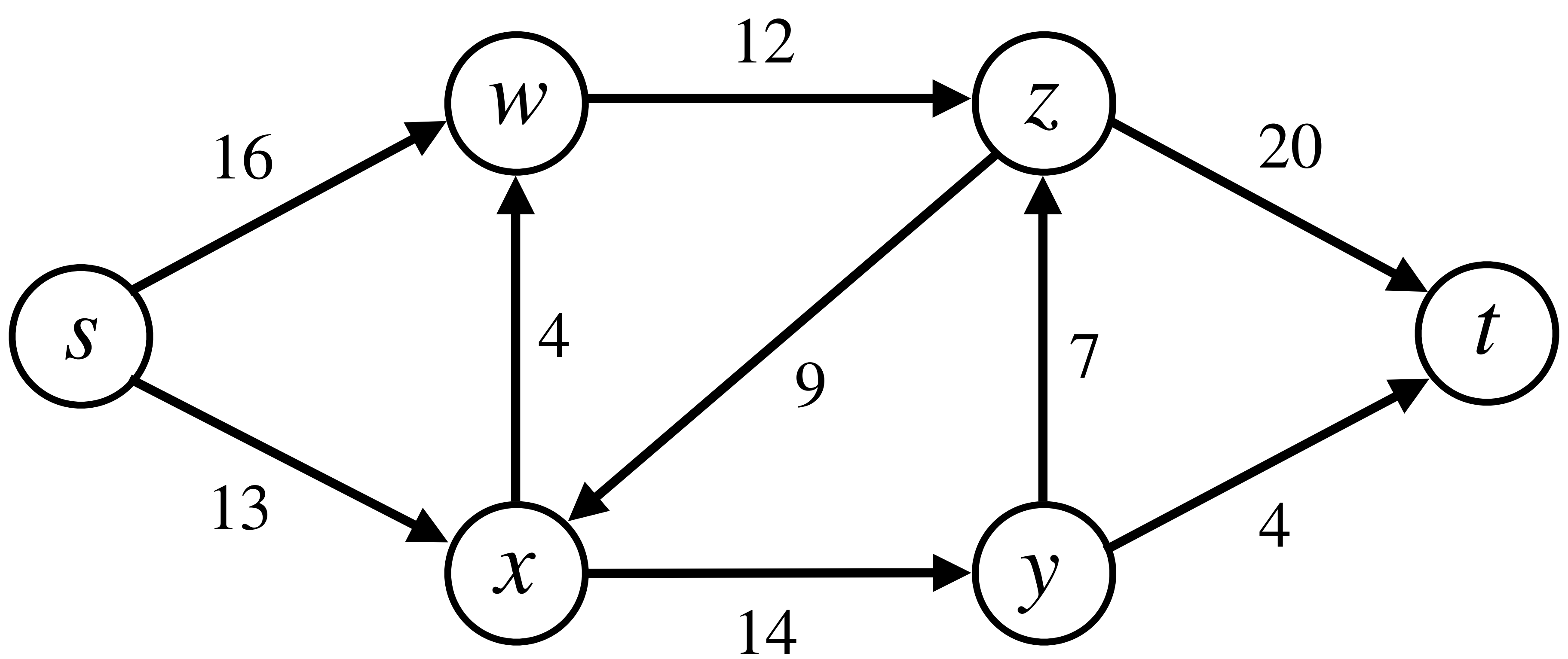

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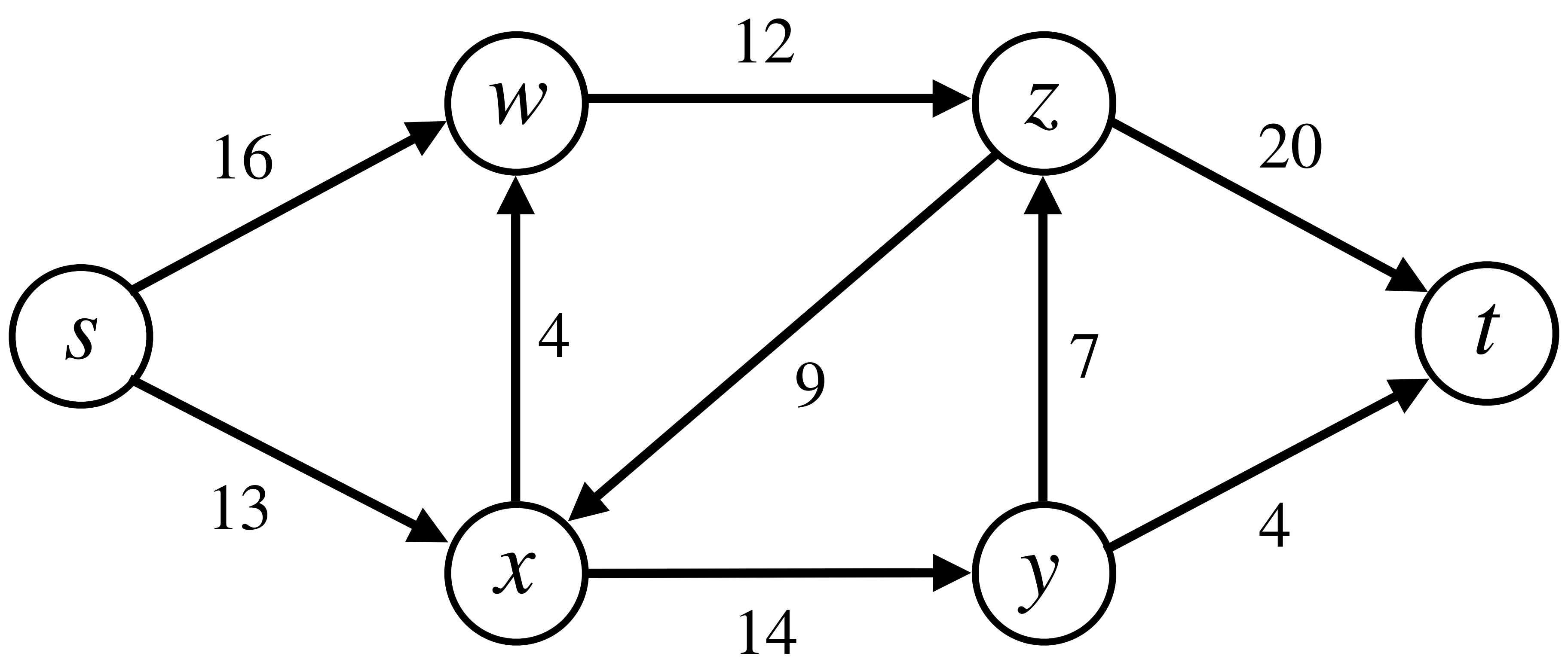

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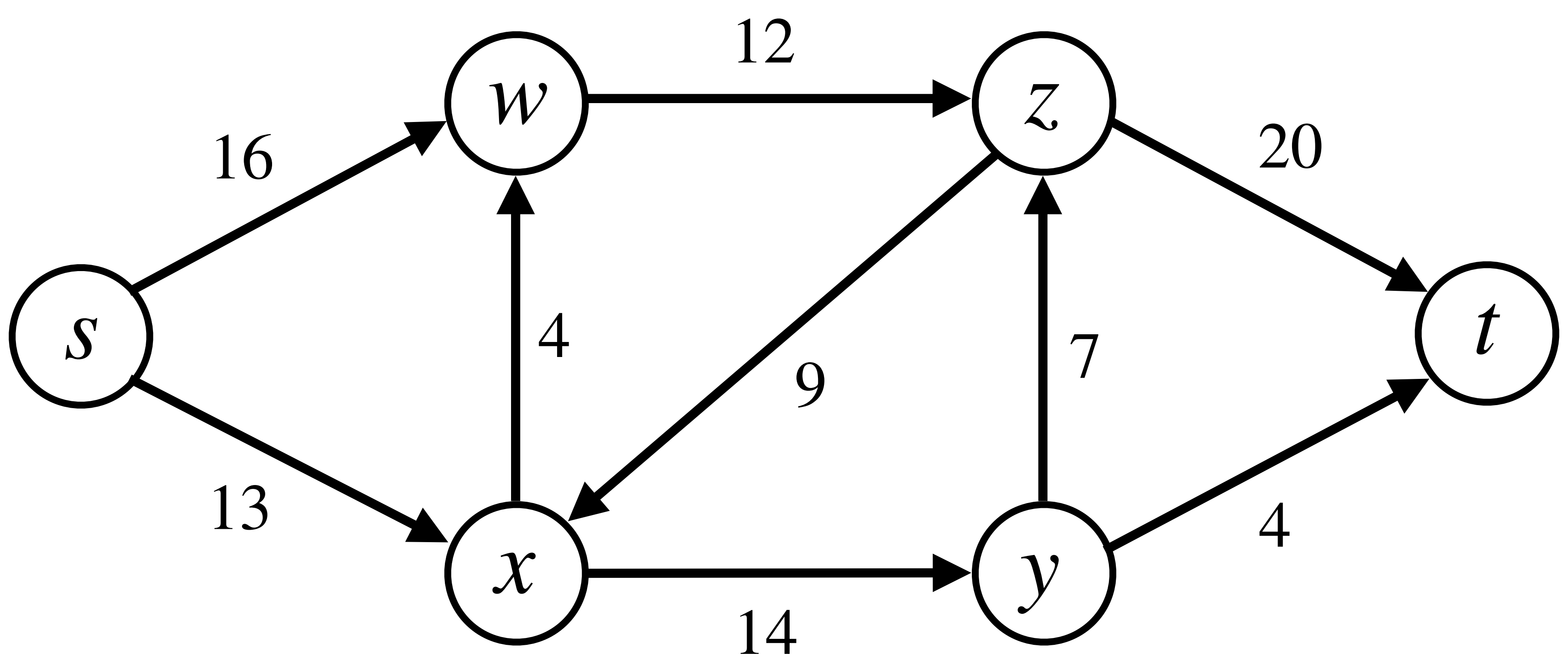

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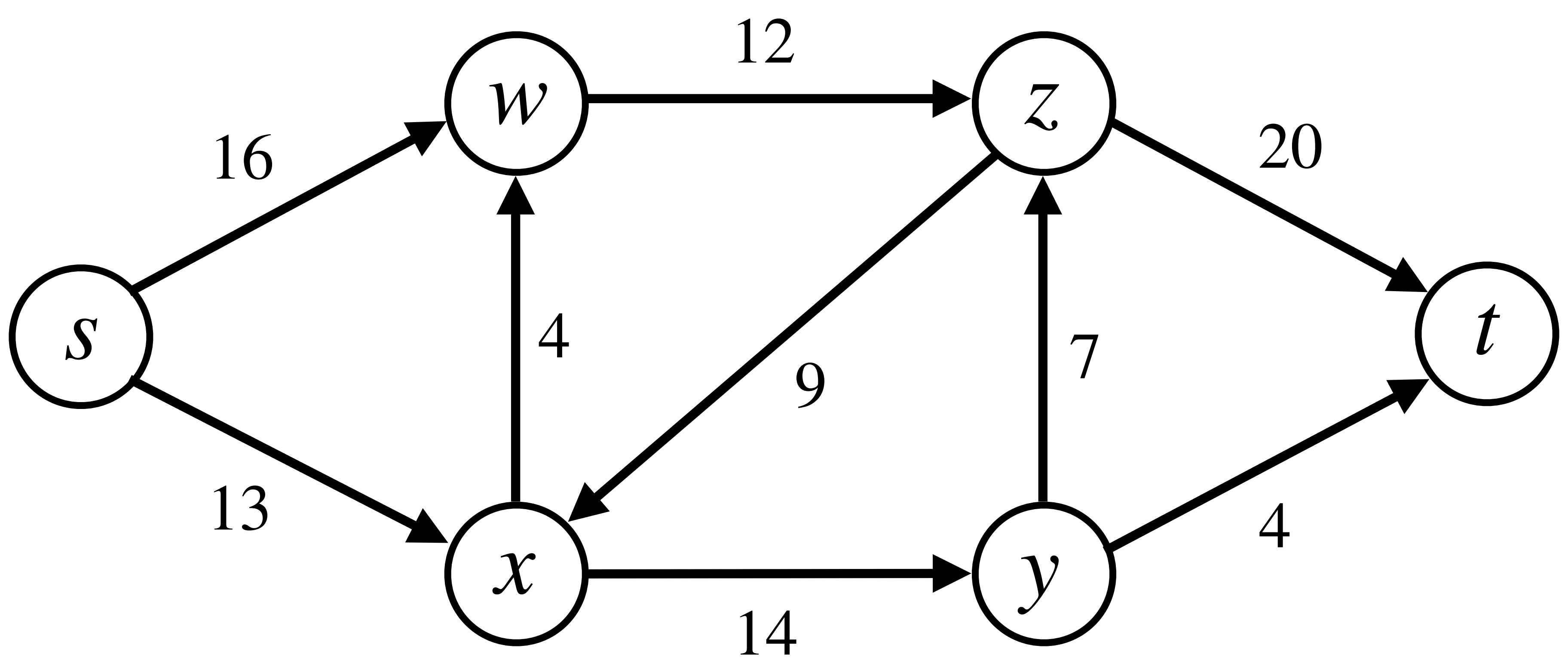

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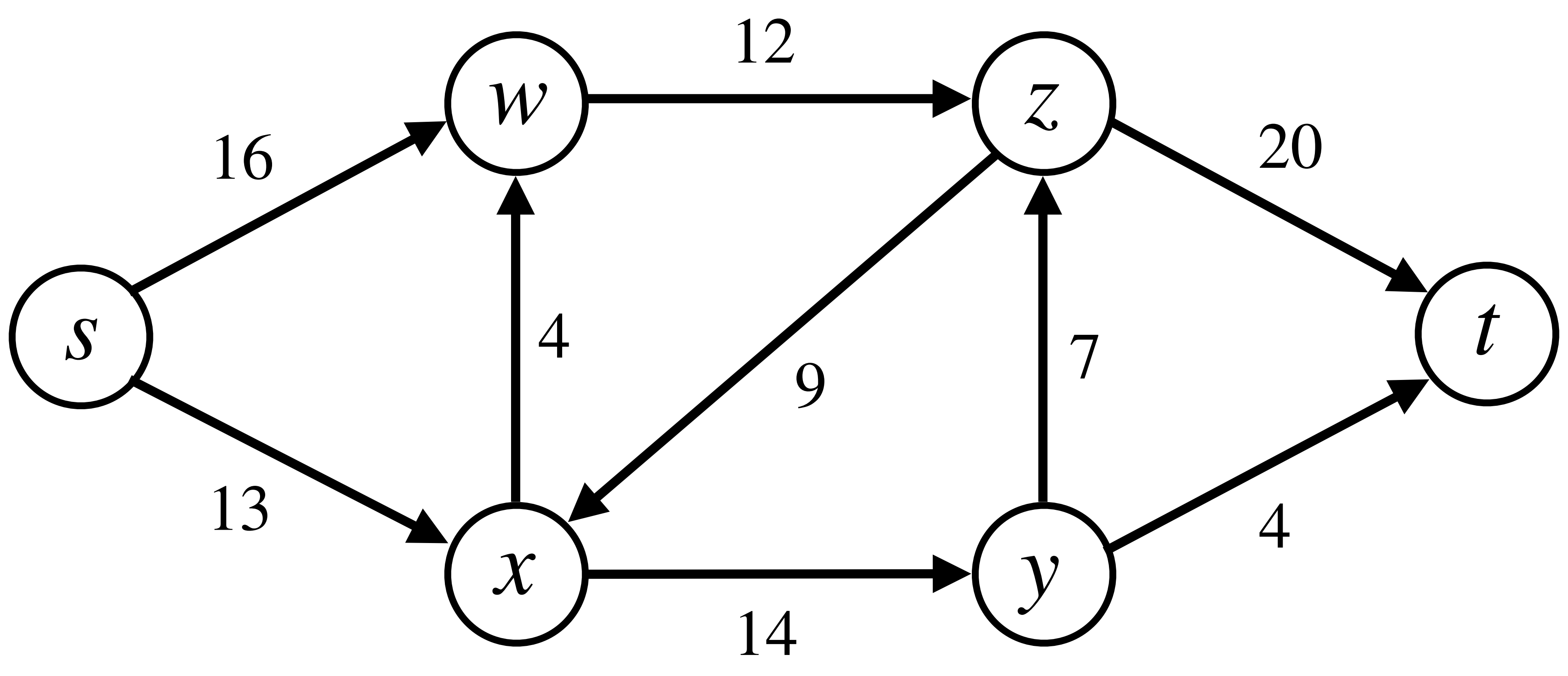

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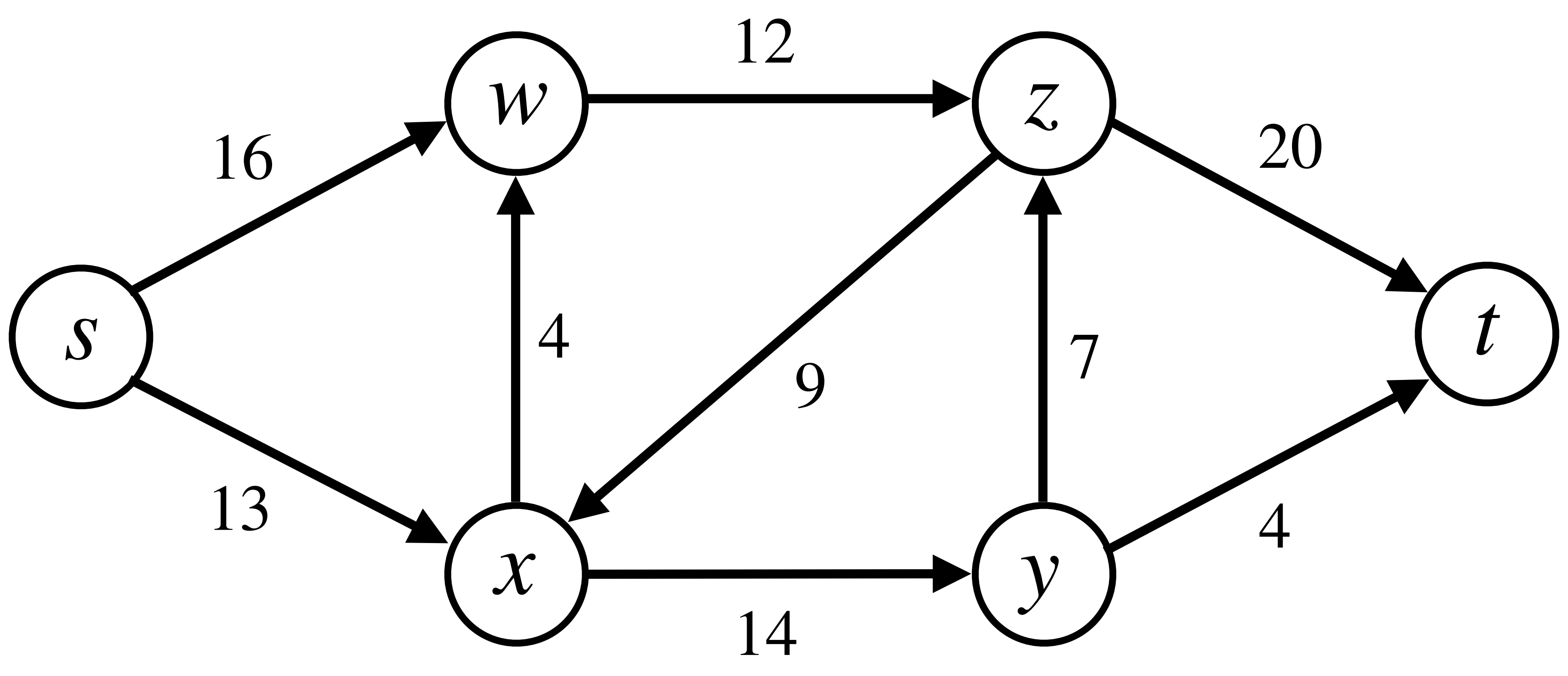

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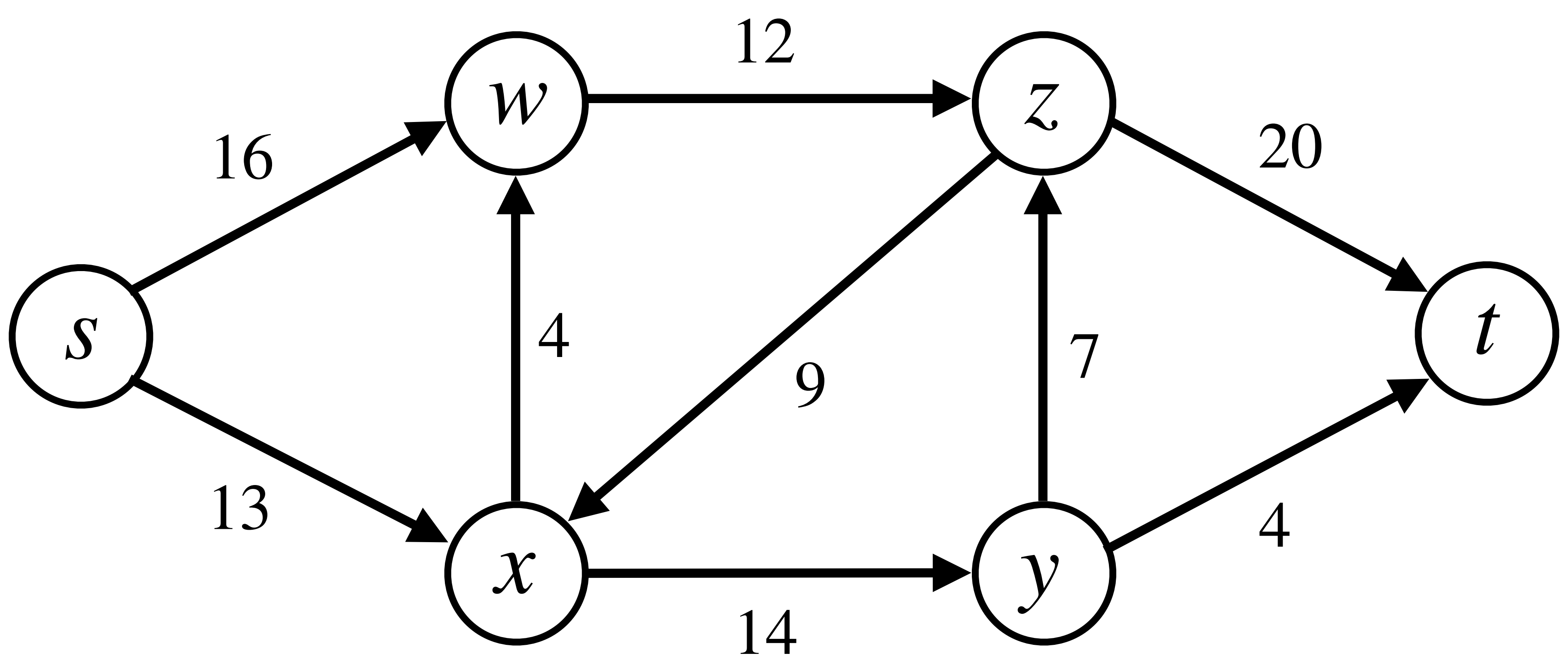
- Two distinguished vertices: **source** s (no incoming edges) and **sink** t (no outgoing edges).
- For every $v \in V$, some $s \rightsquigarrow v \rightsquigarrow t$ path exists. Hence, $|E| \geq |V| - 1$.



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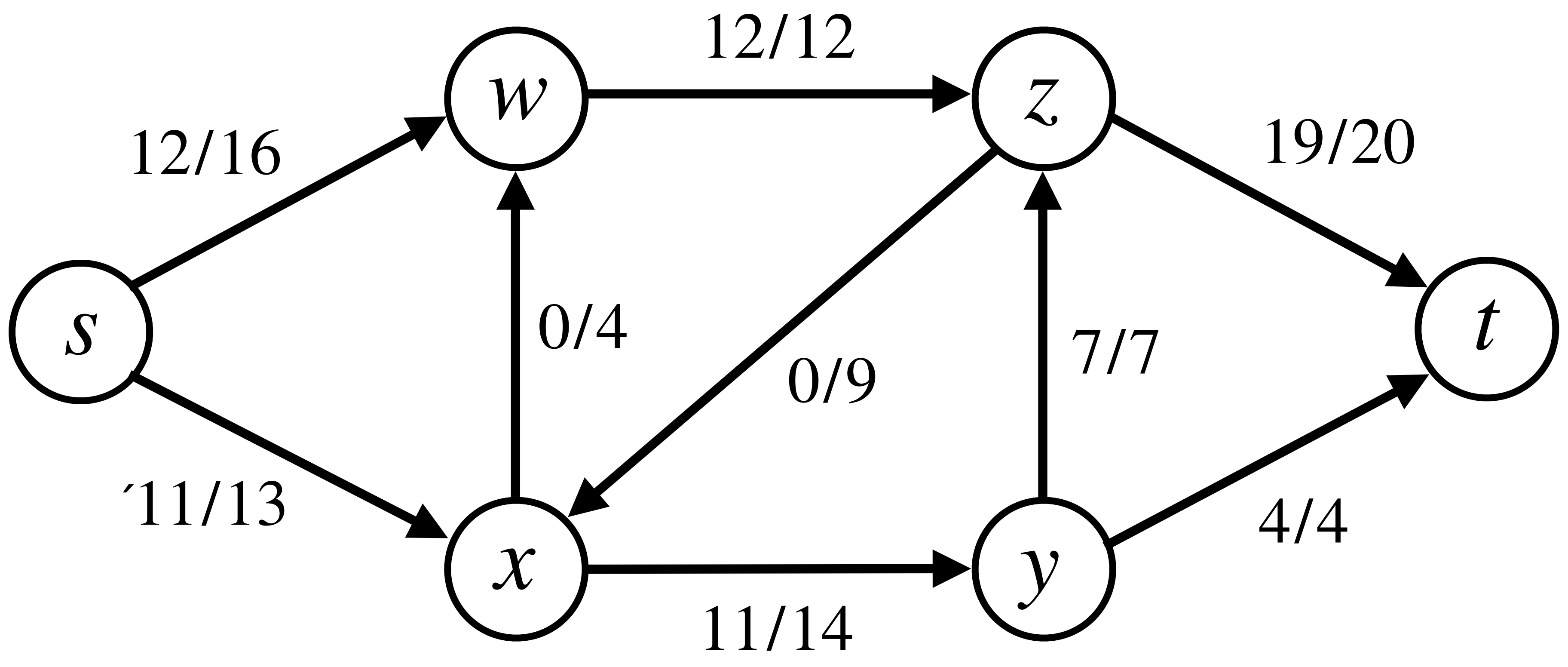

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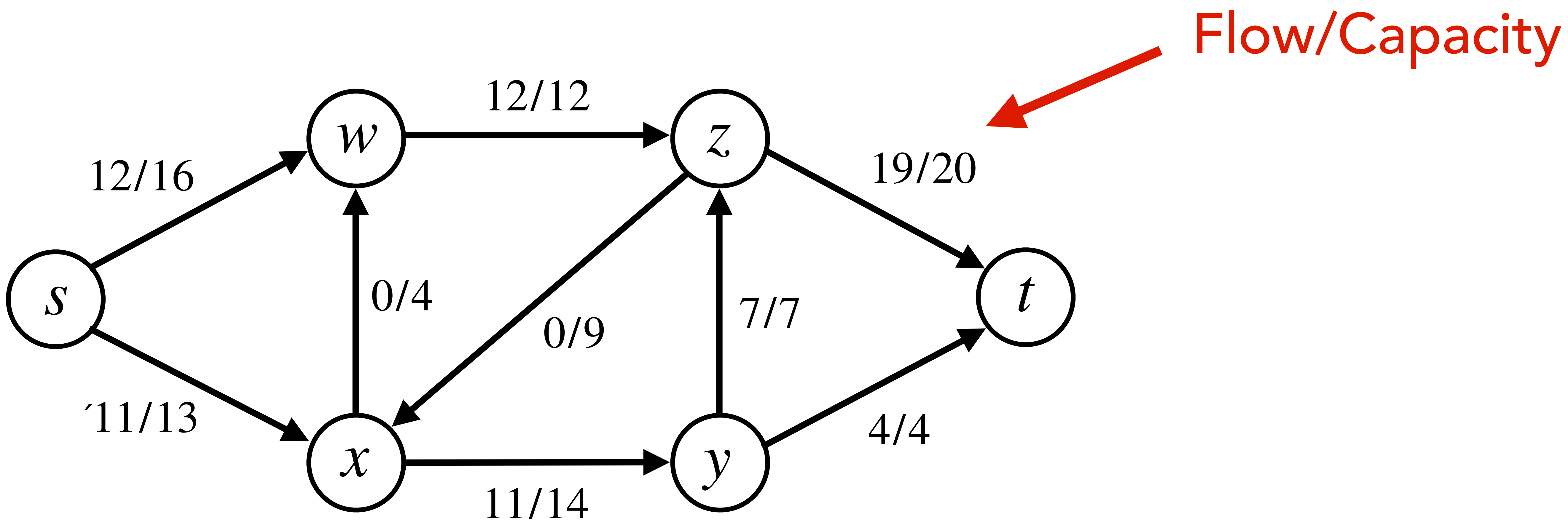

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Output: Flow of maximum value.

Finding Max Flow: An Attempt

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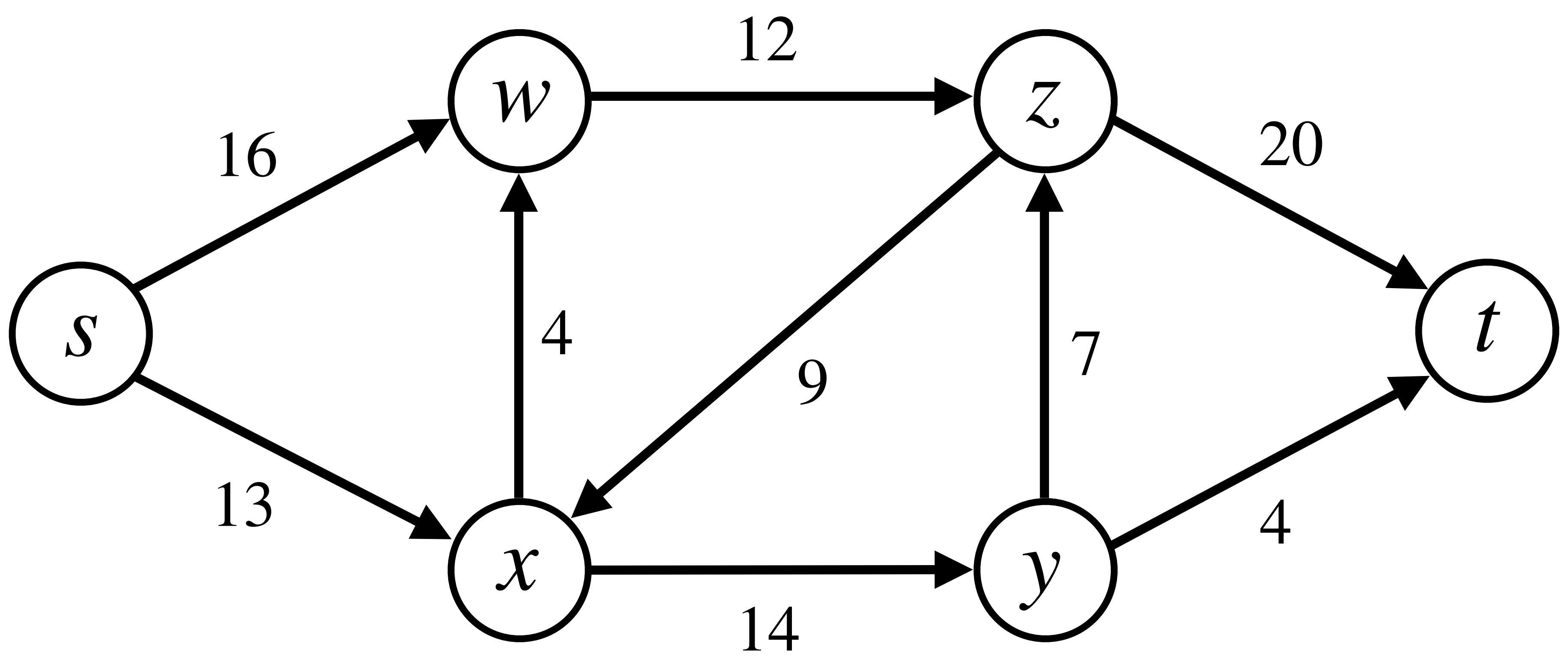

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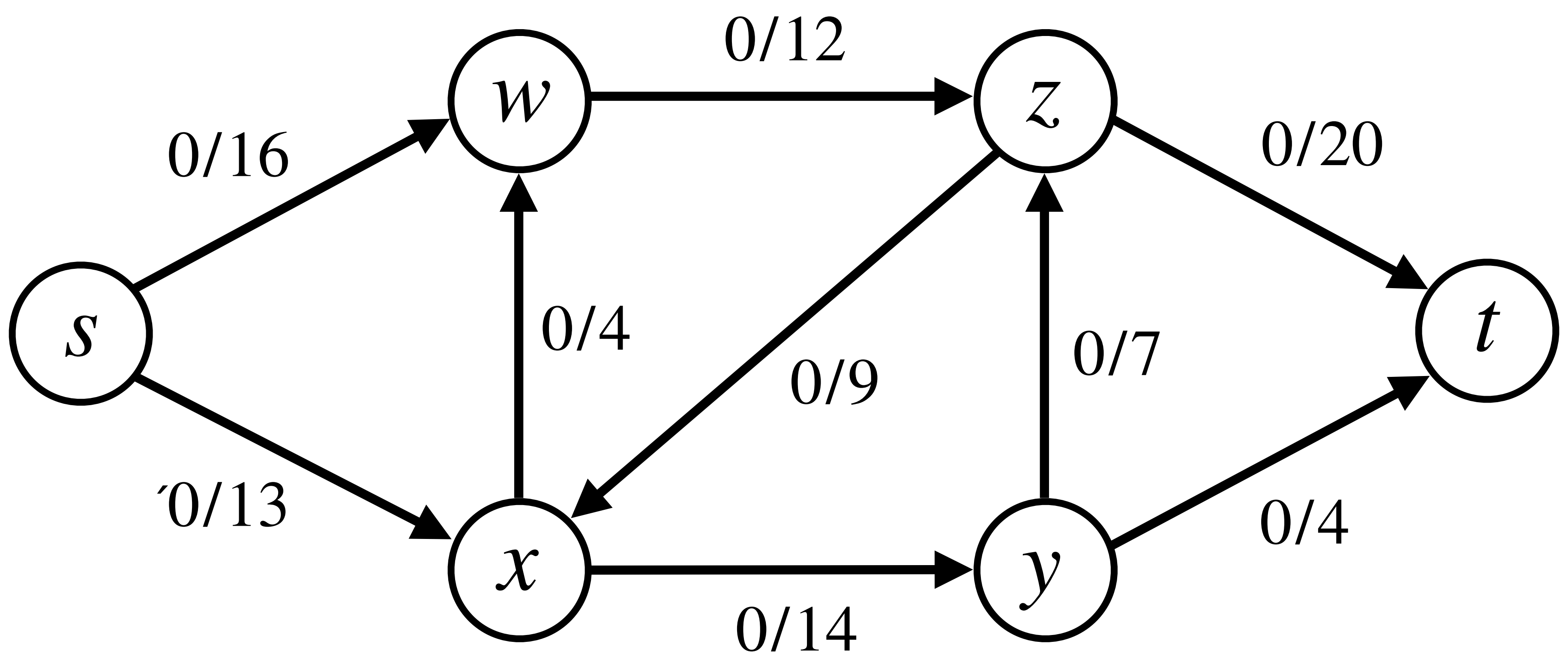

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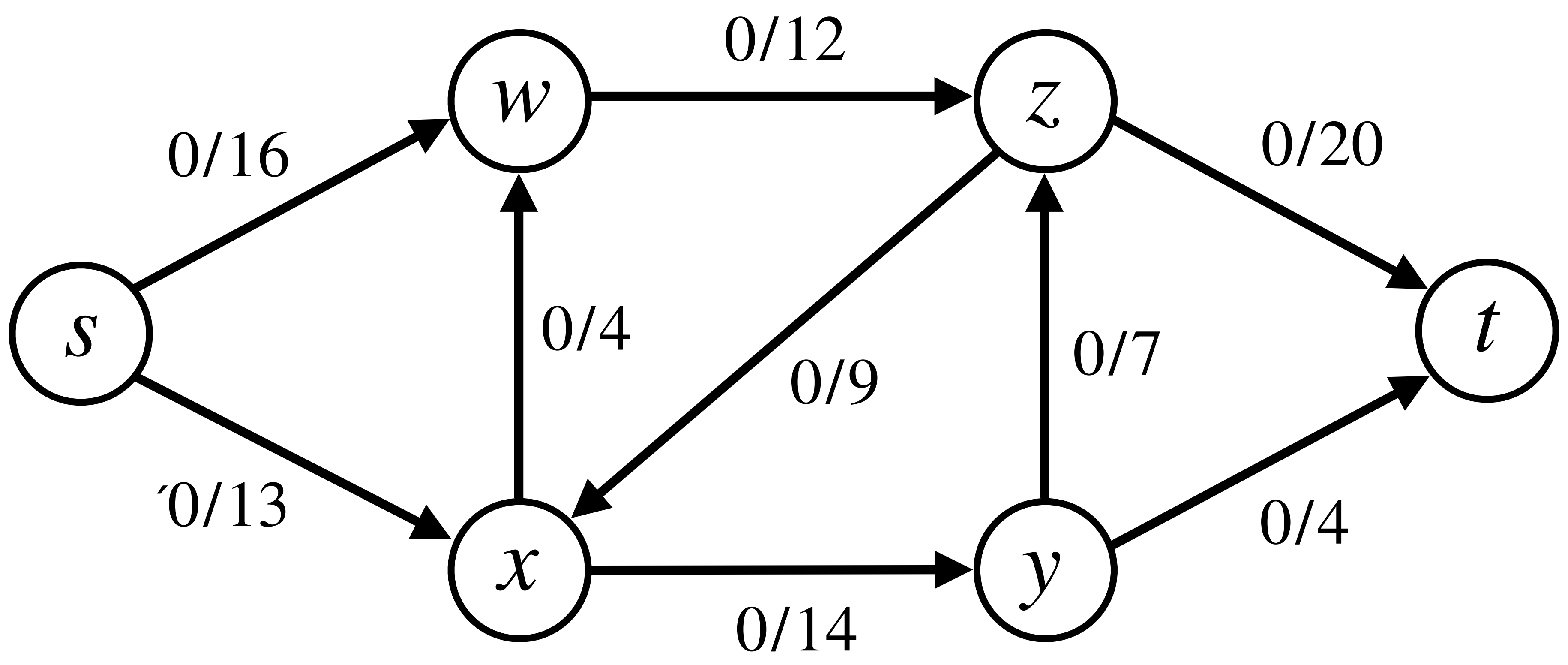

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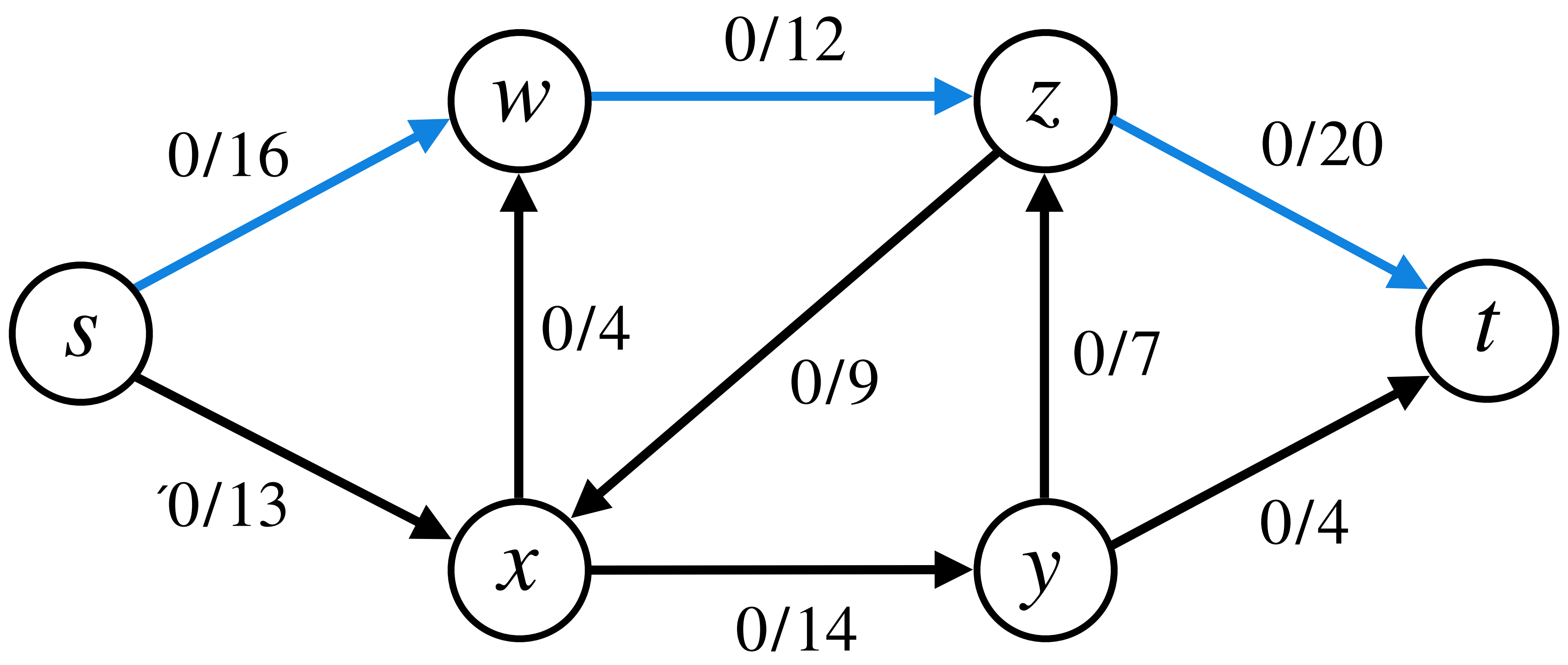

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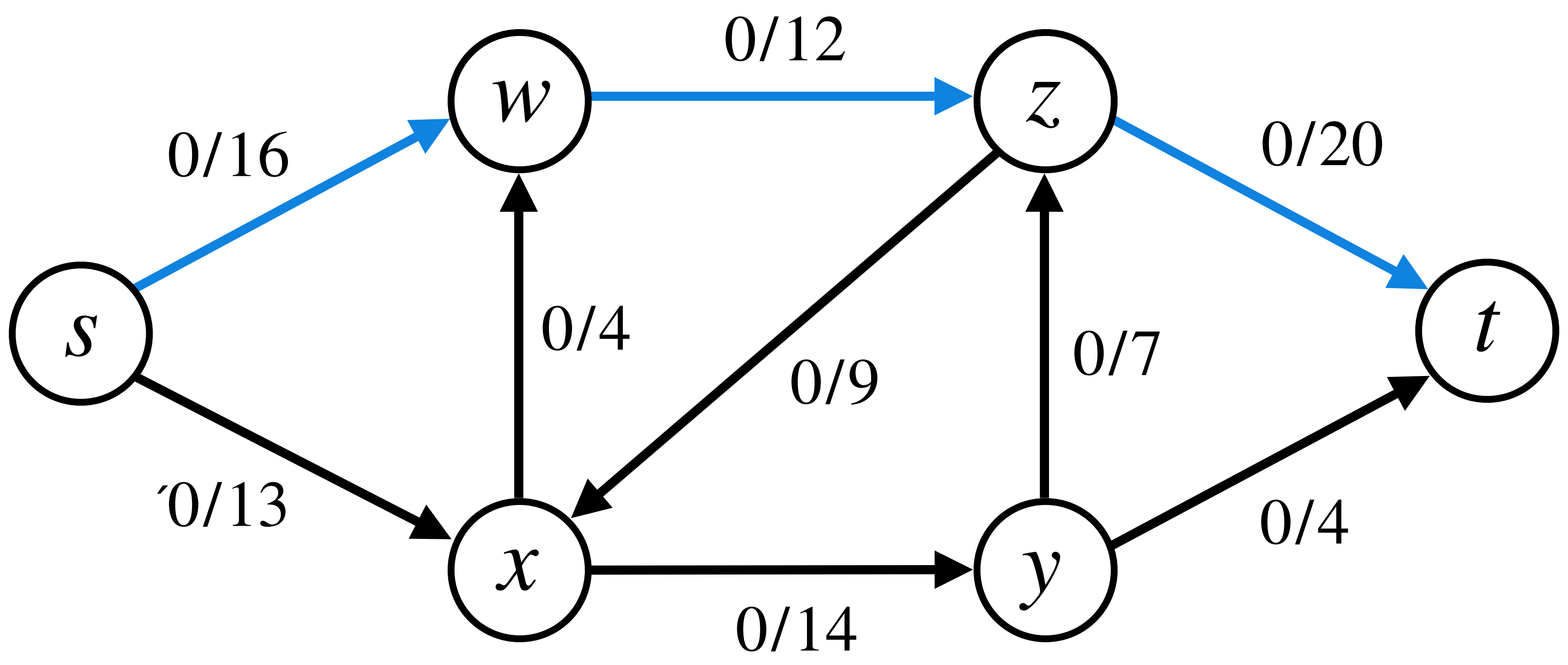

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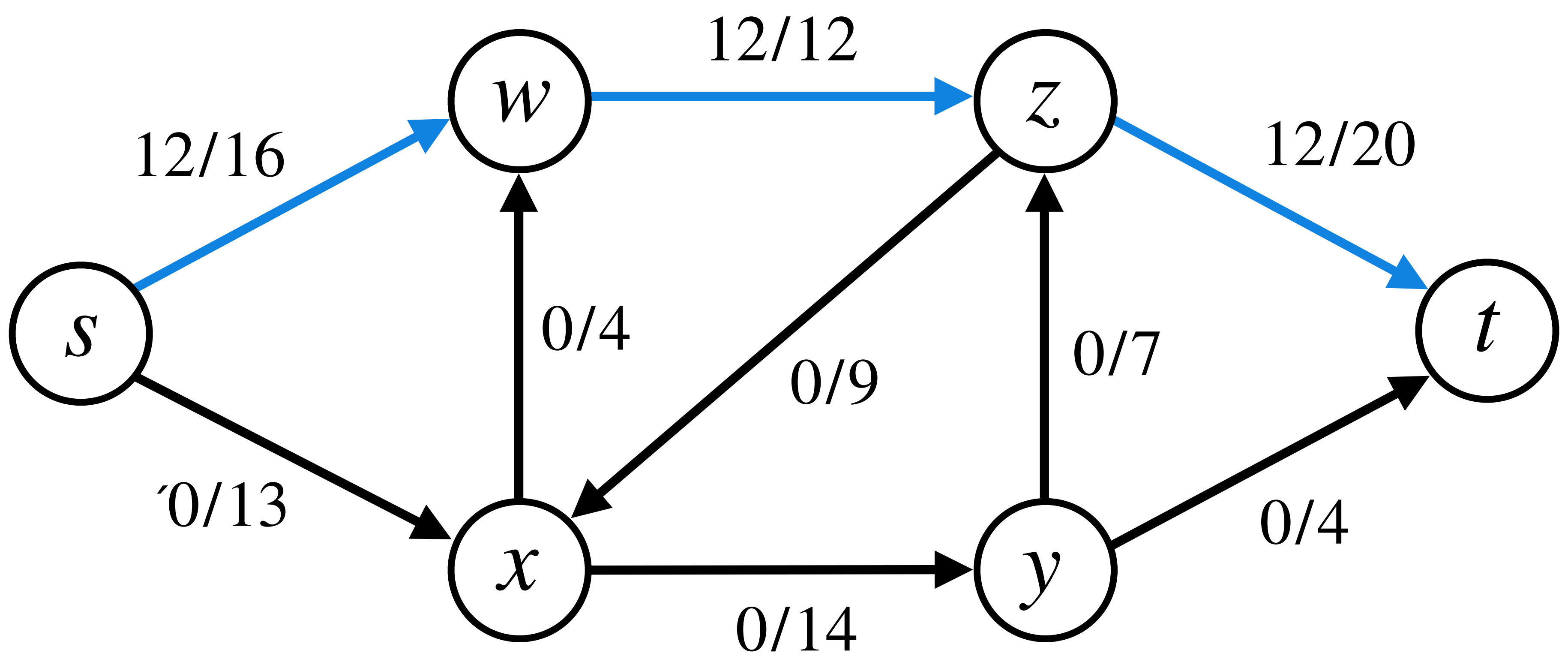

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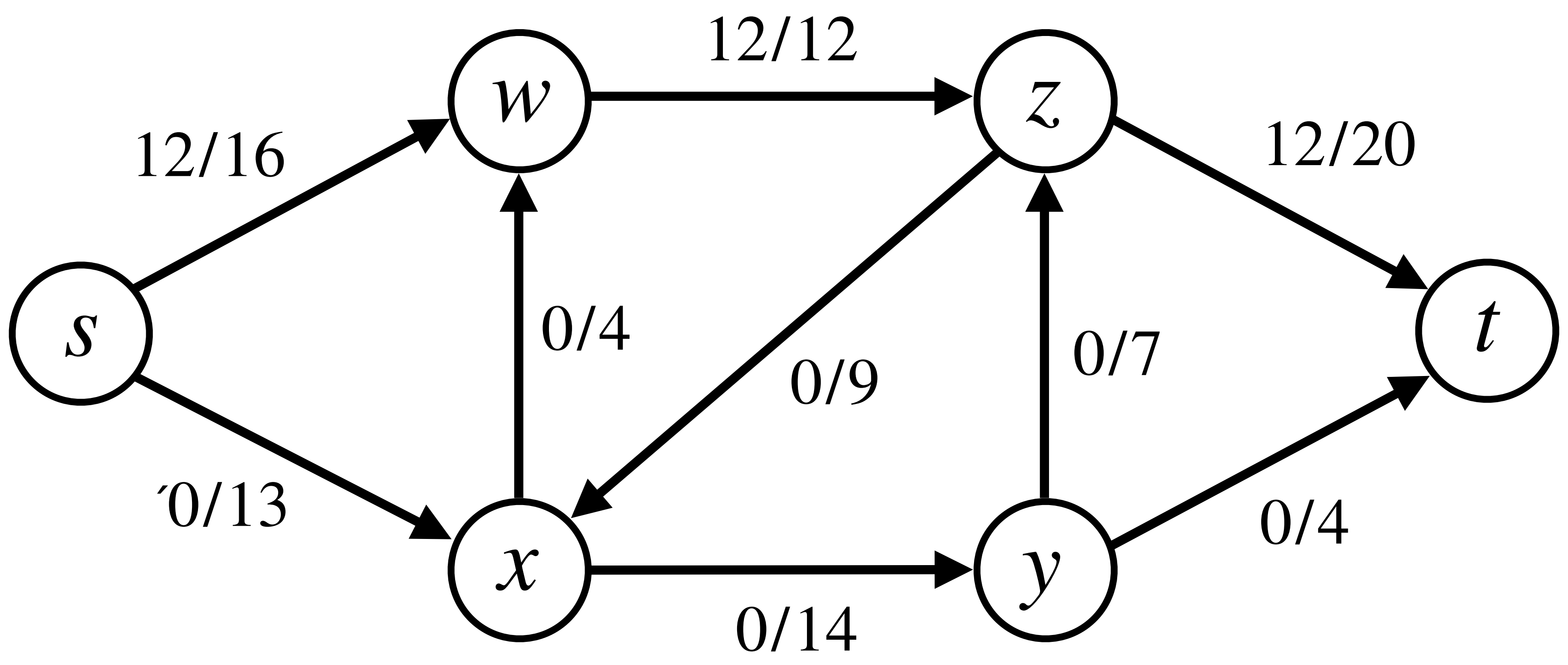

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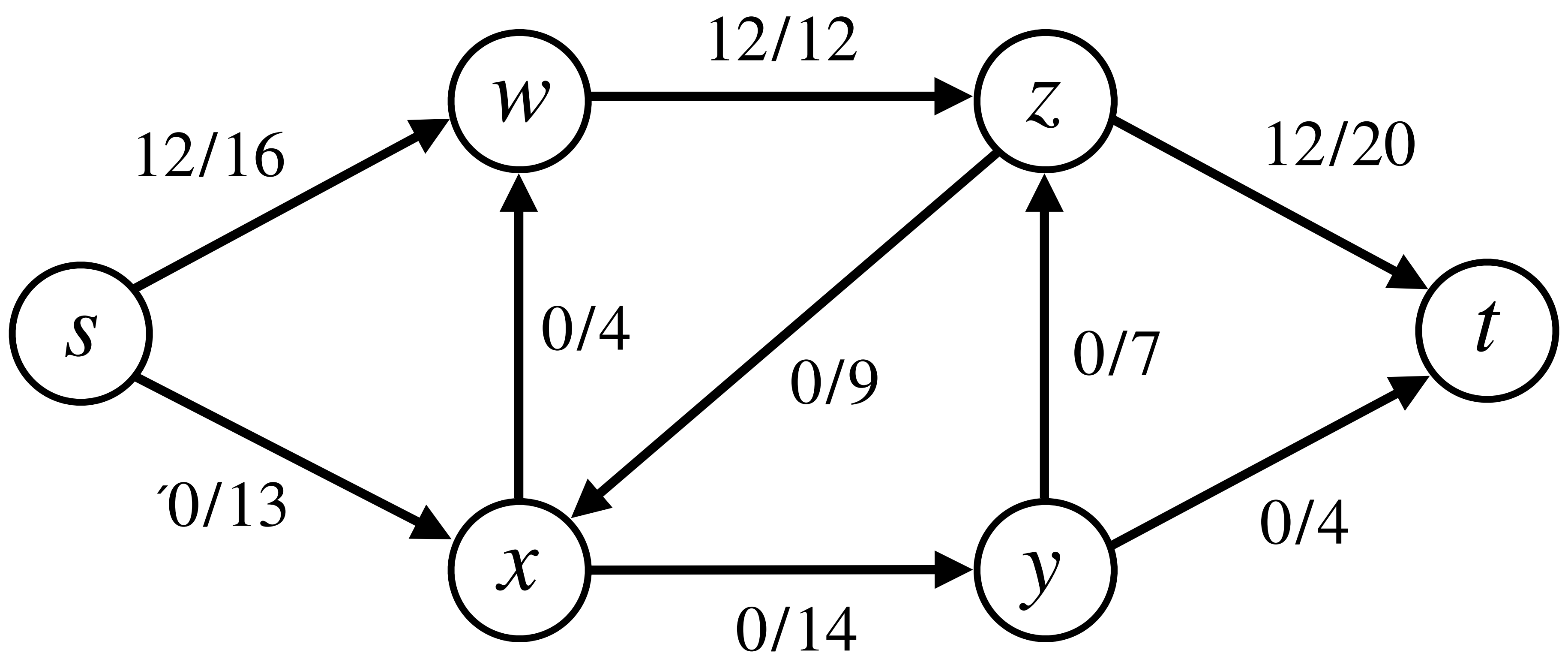

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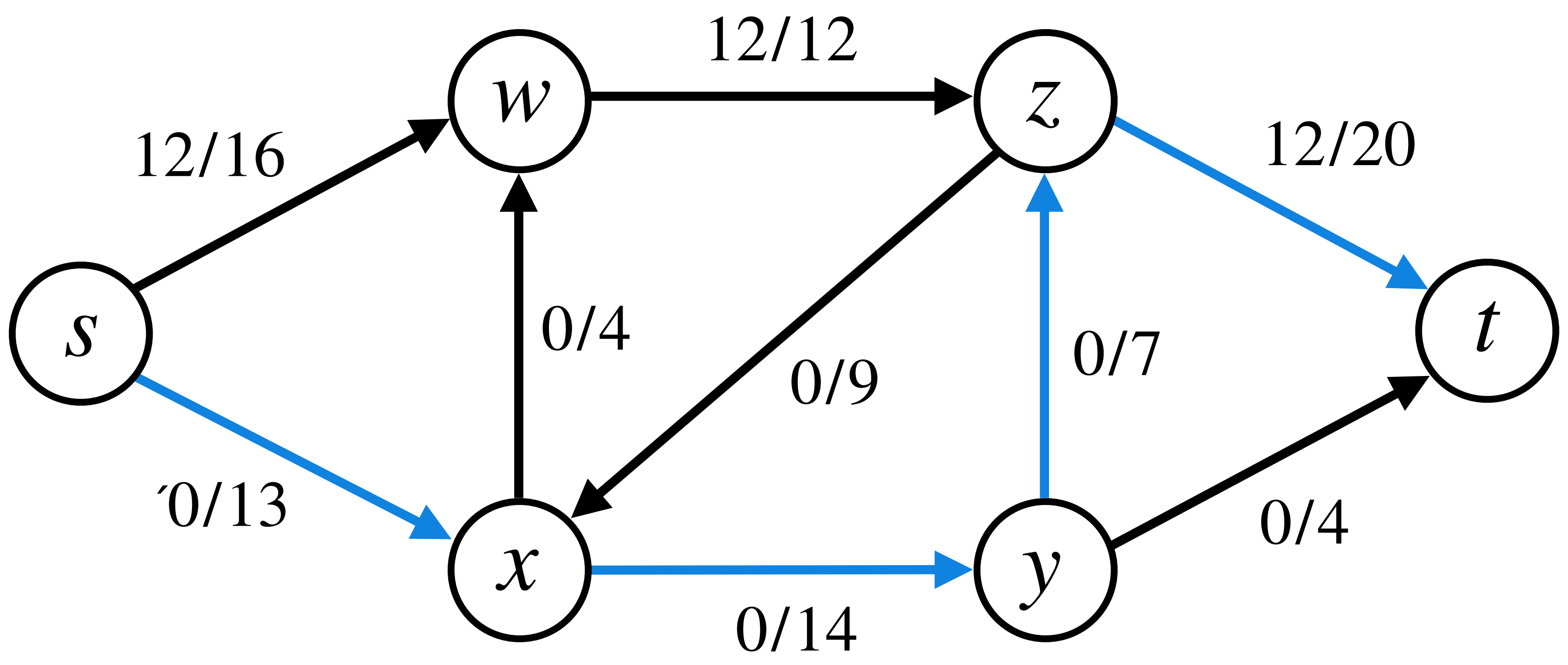

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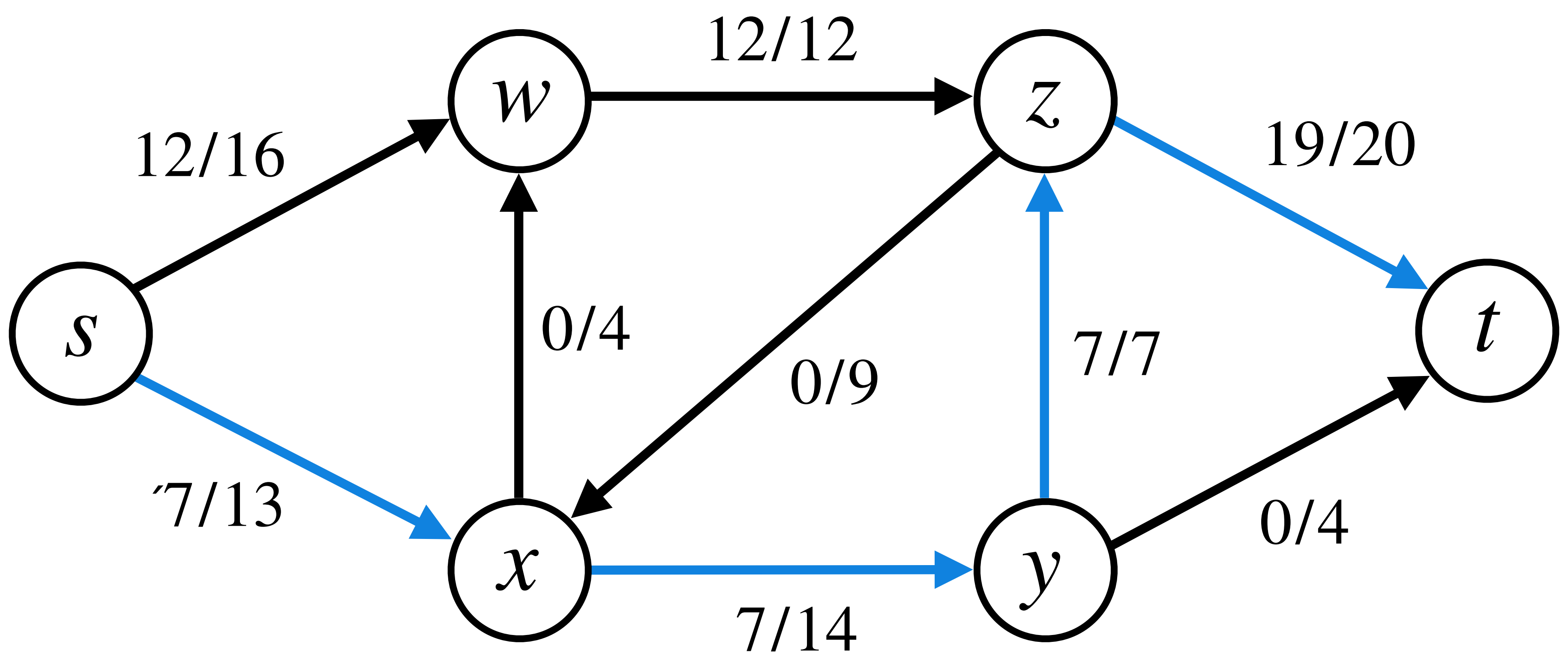

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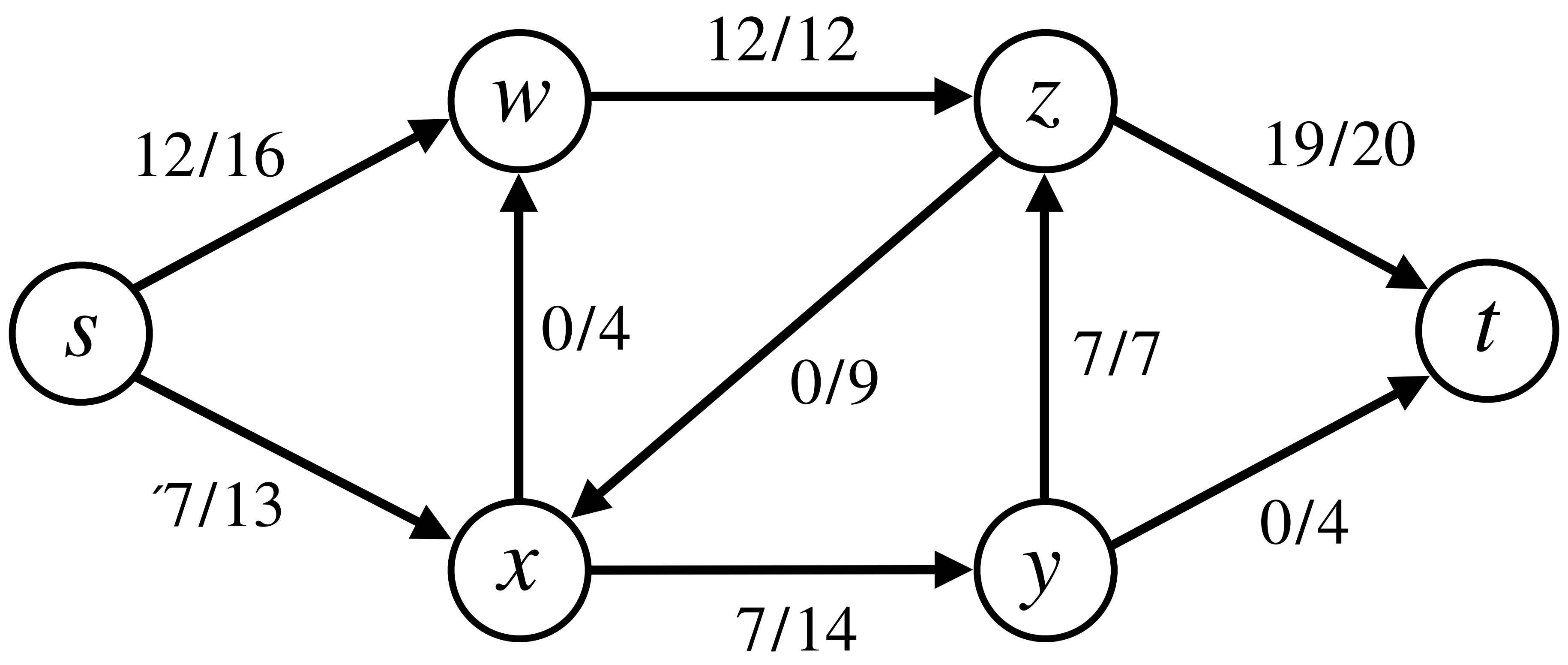

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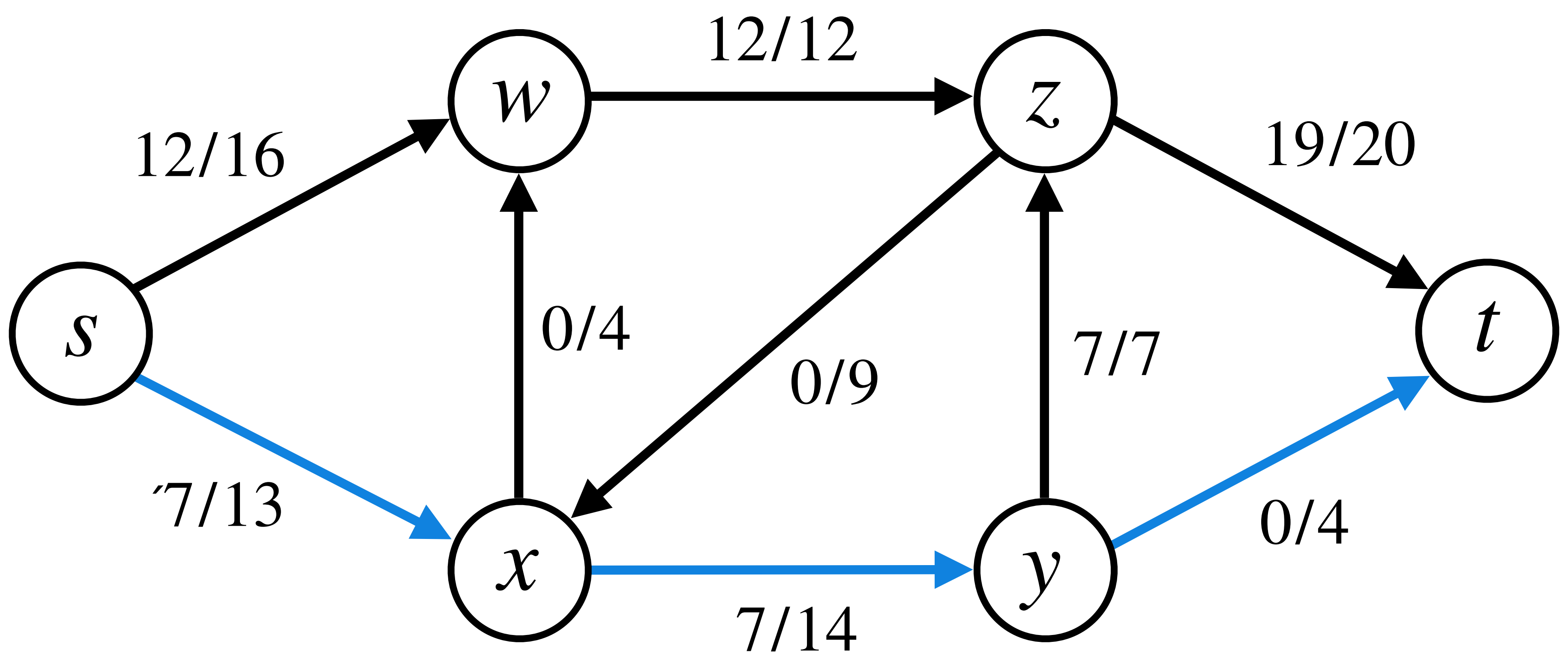

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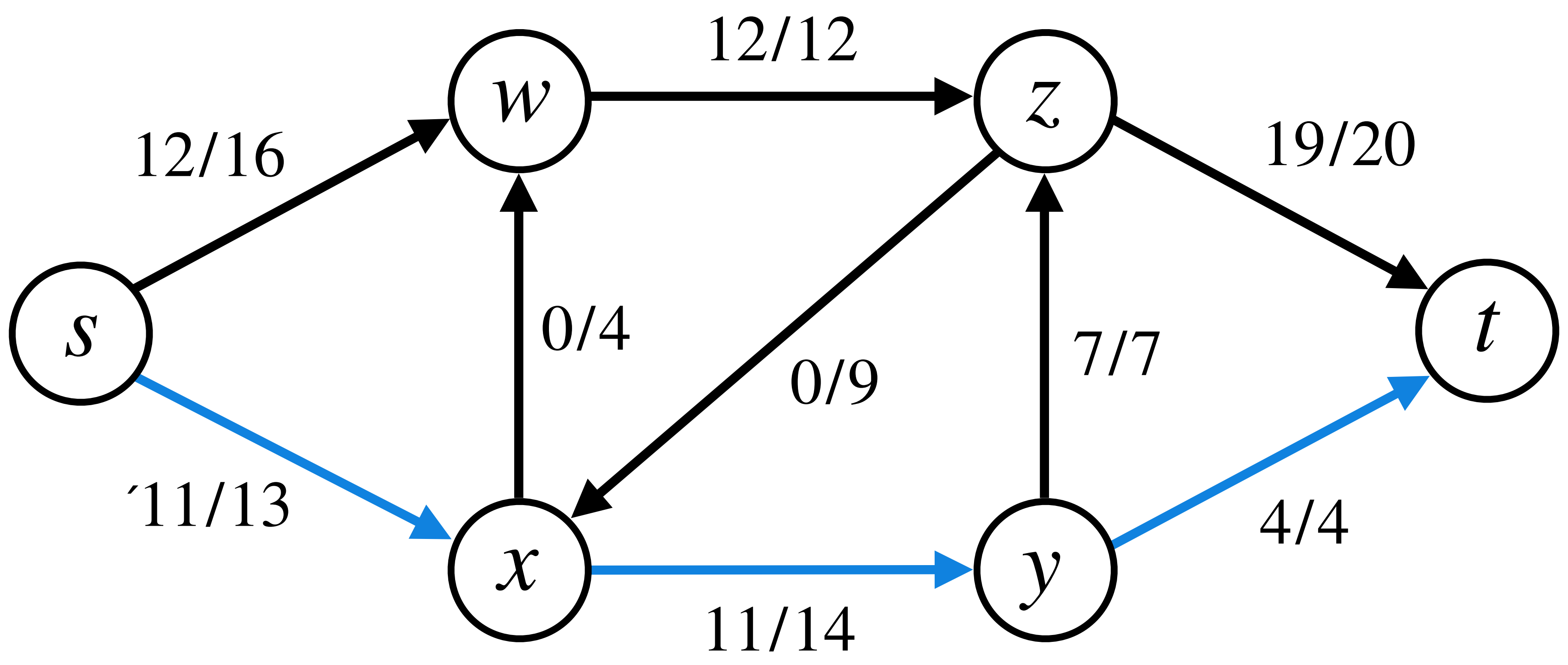

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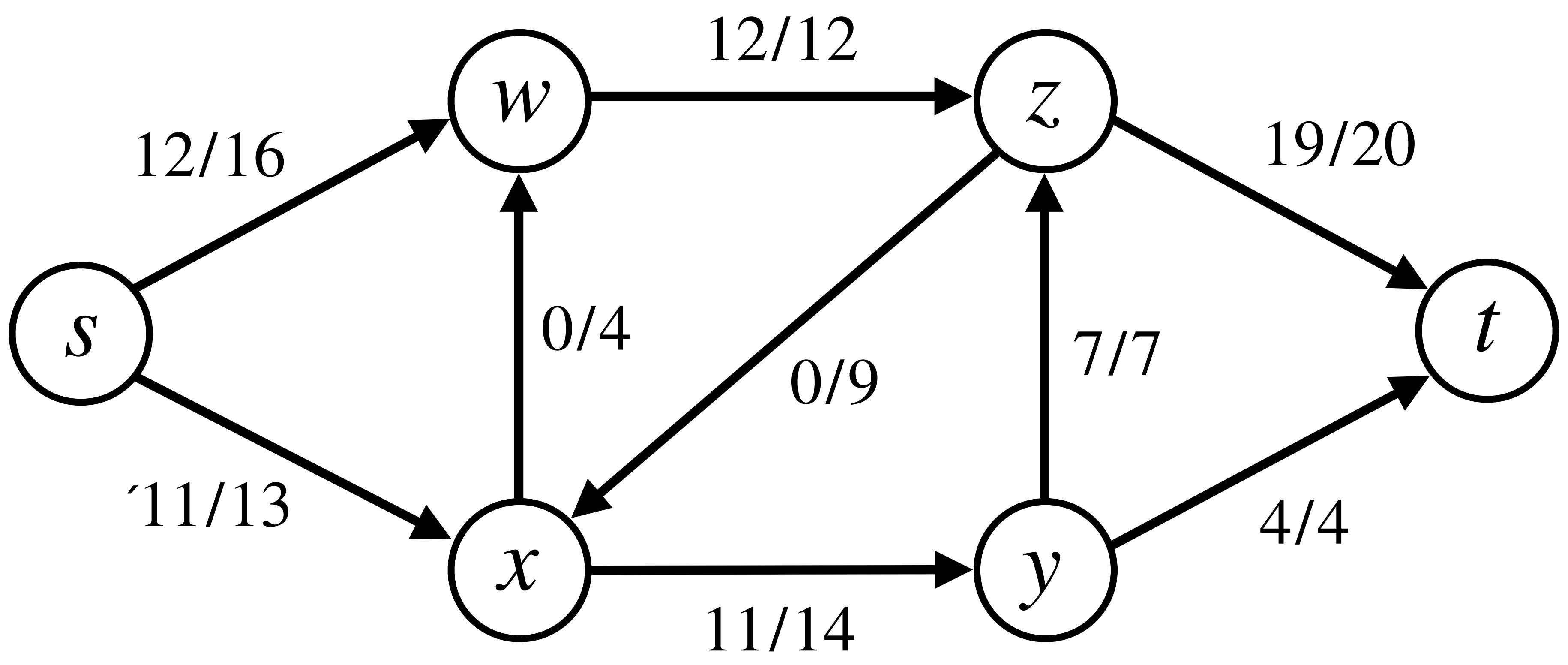

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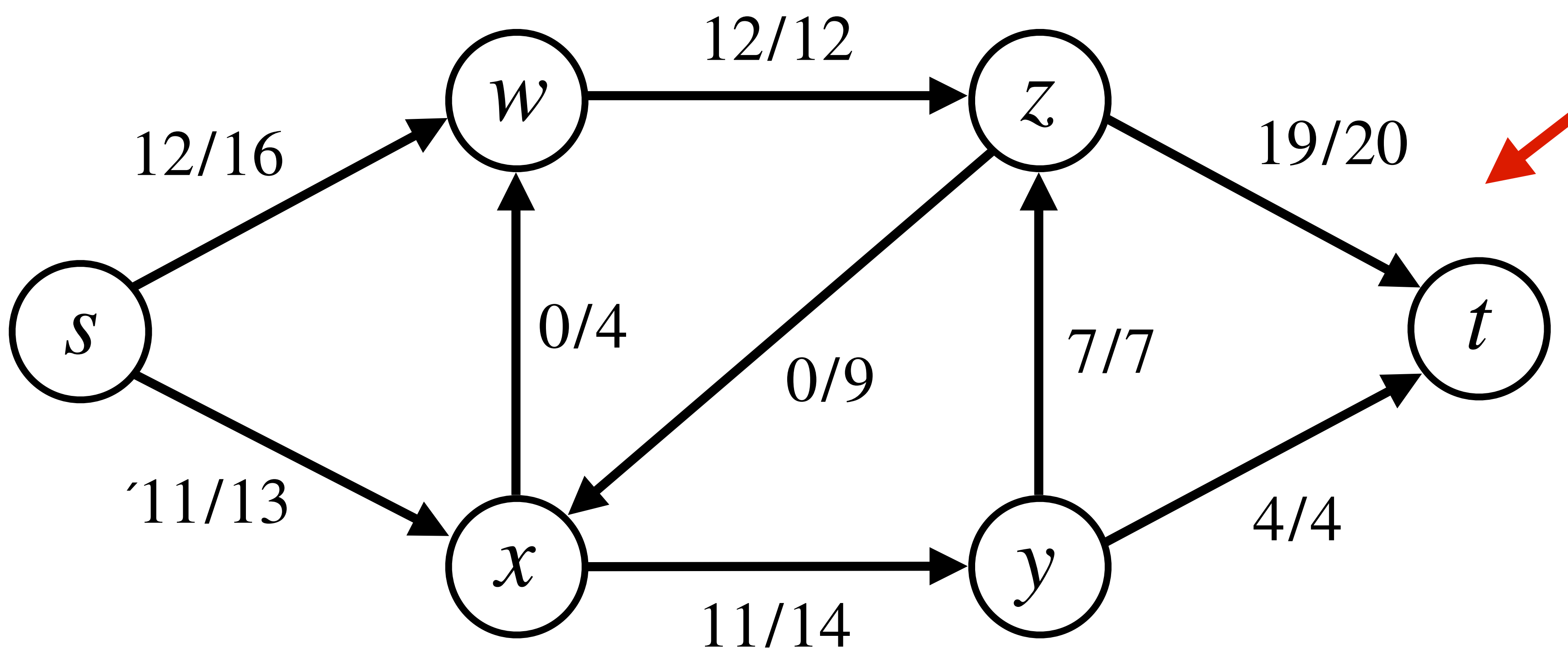

Required $s \rightsquigarrow t$ path not present

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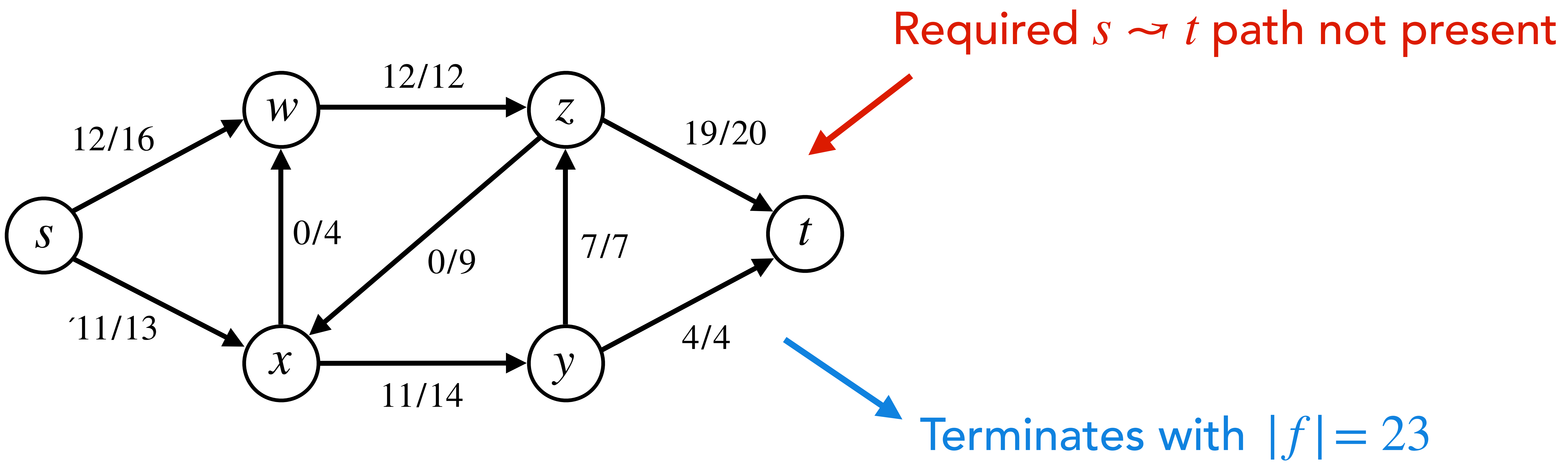

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Is the algorithm really correct?

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Is the algorithm really correct? No.

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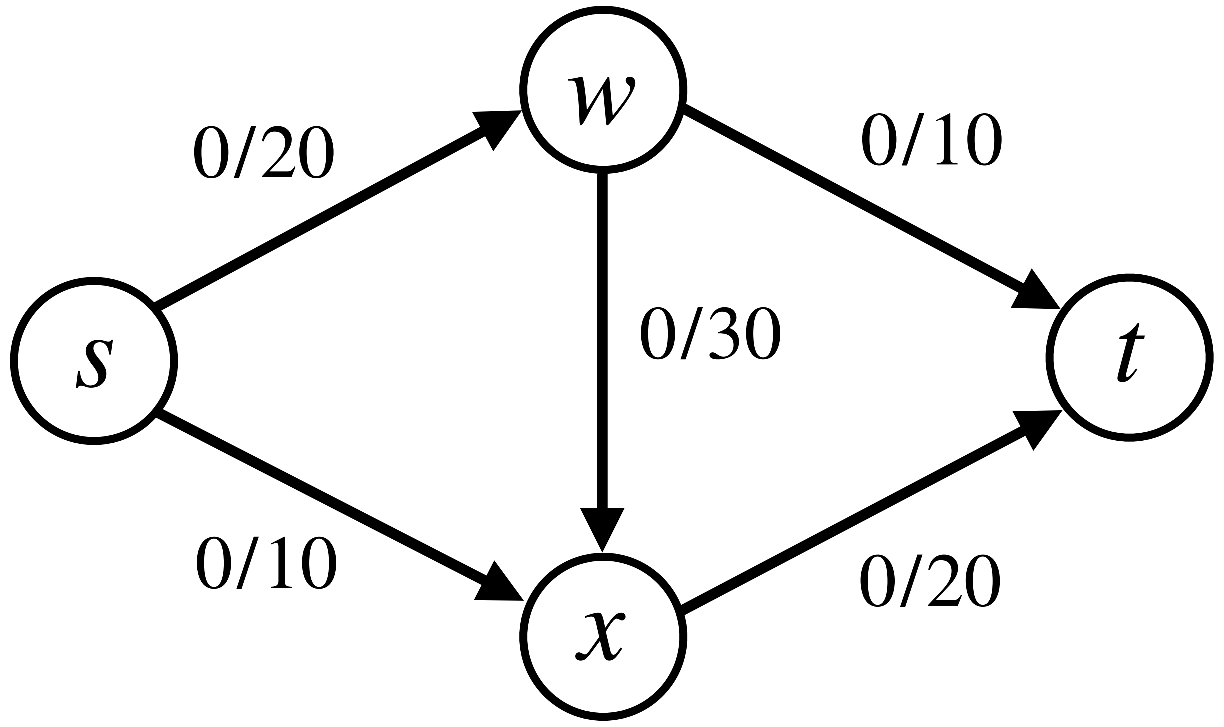

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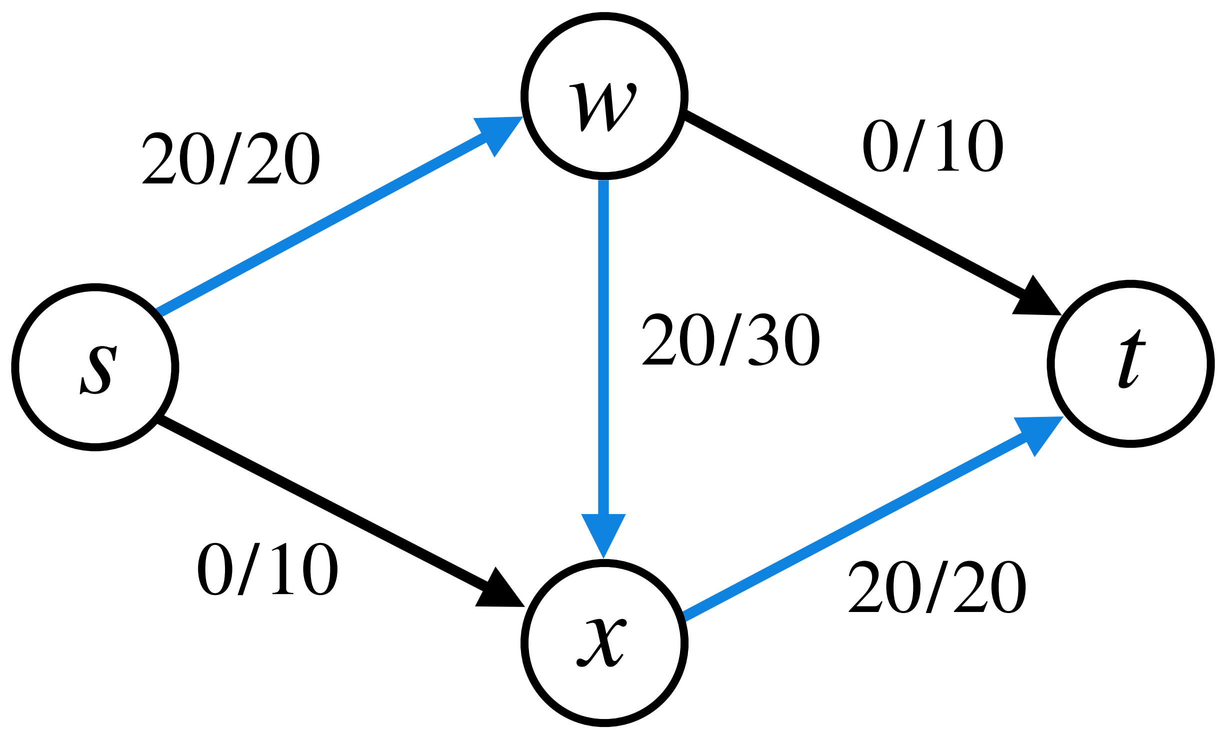

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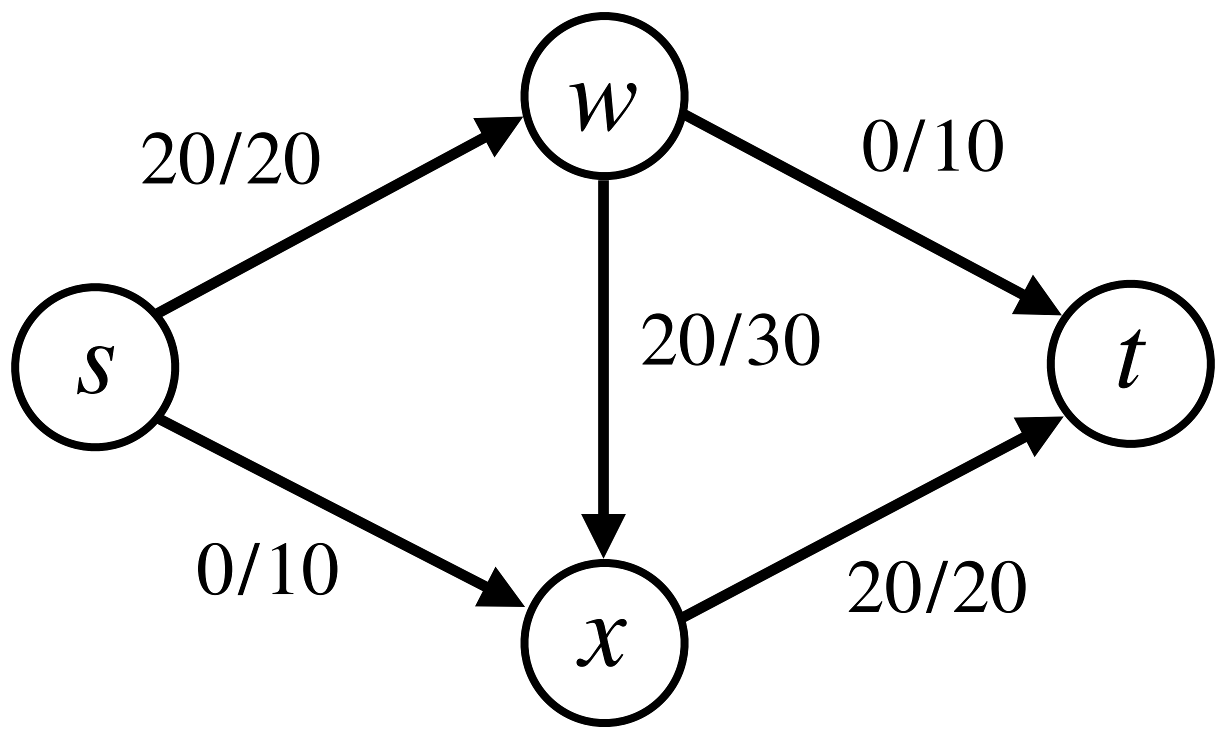

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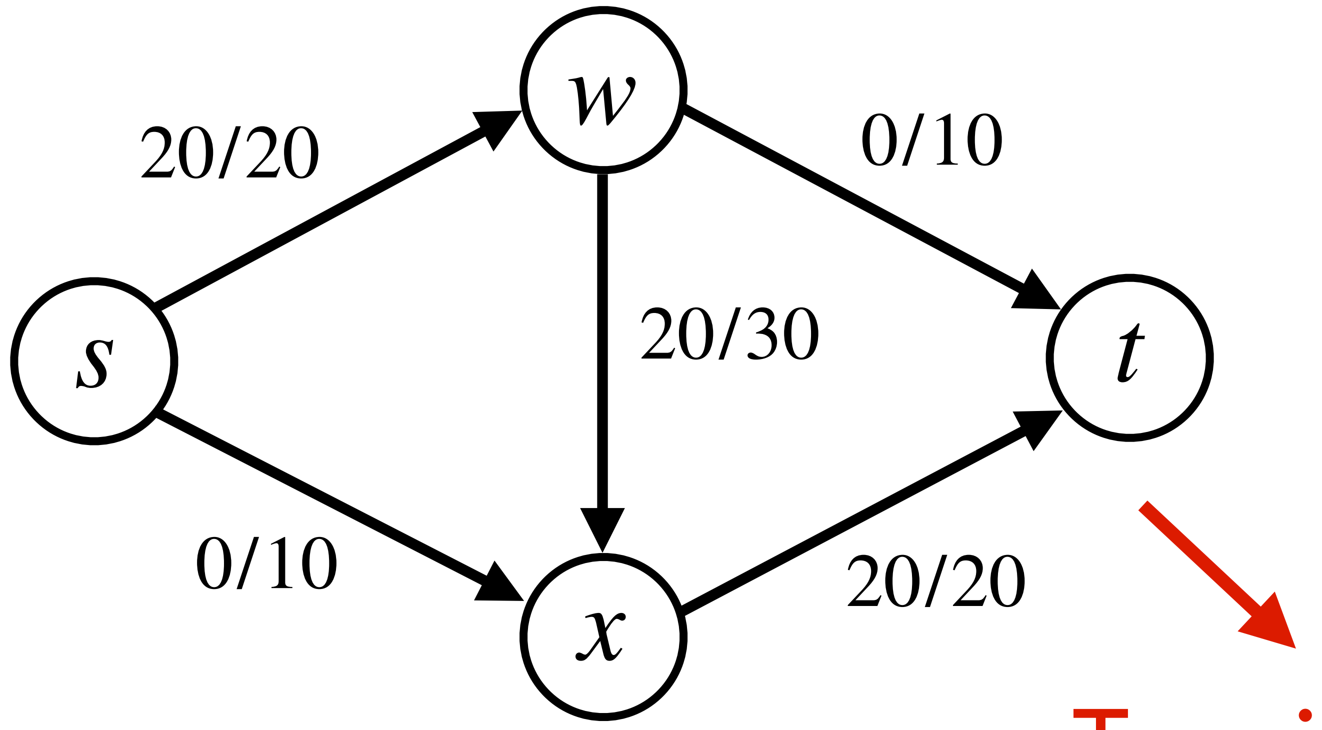

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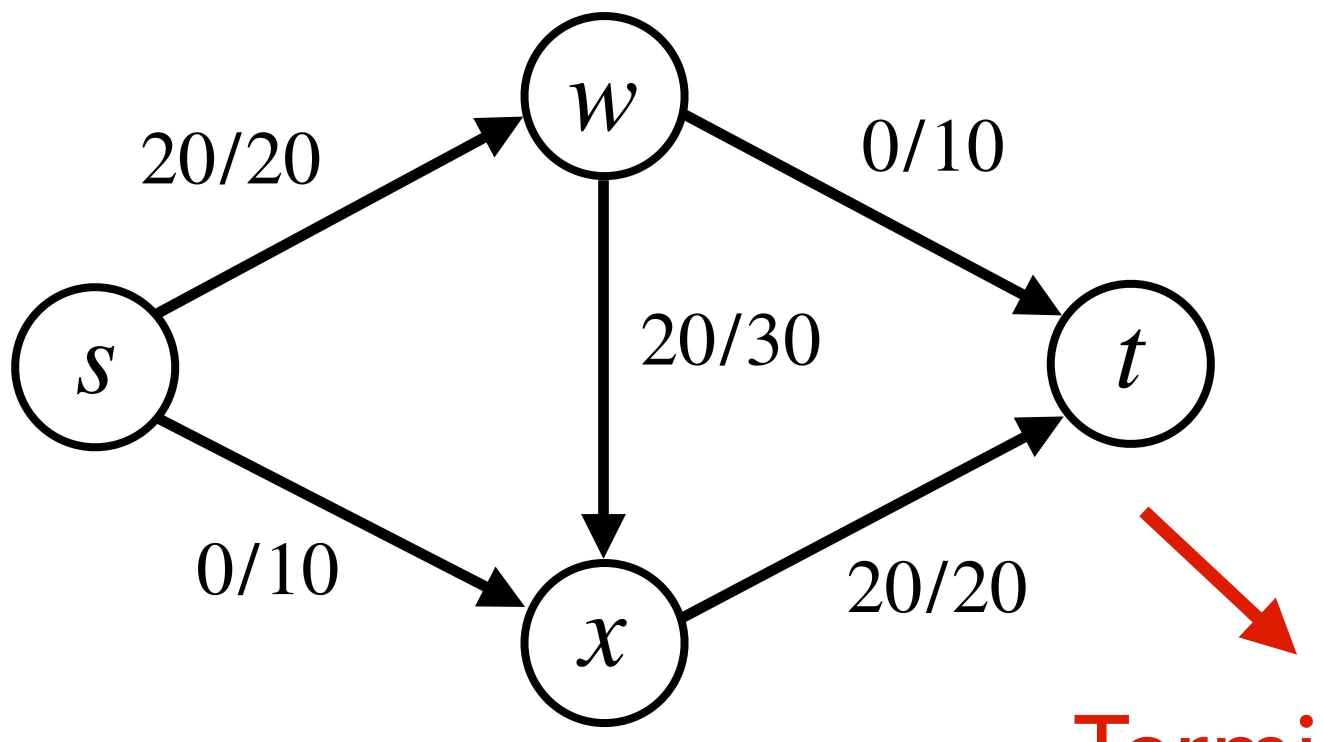

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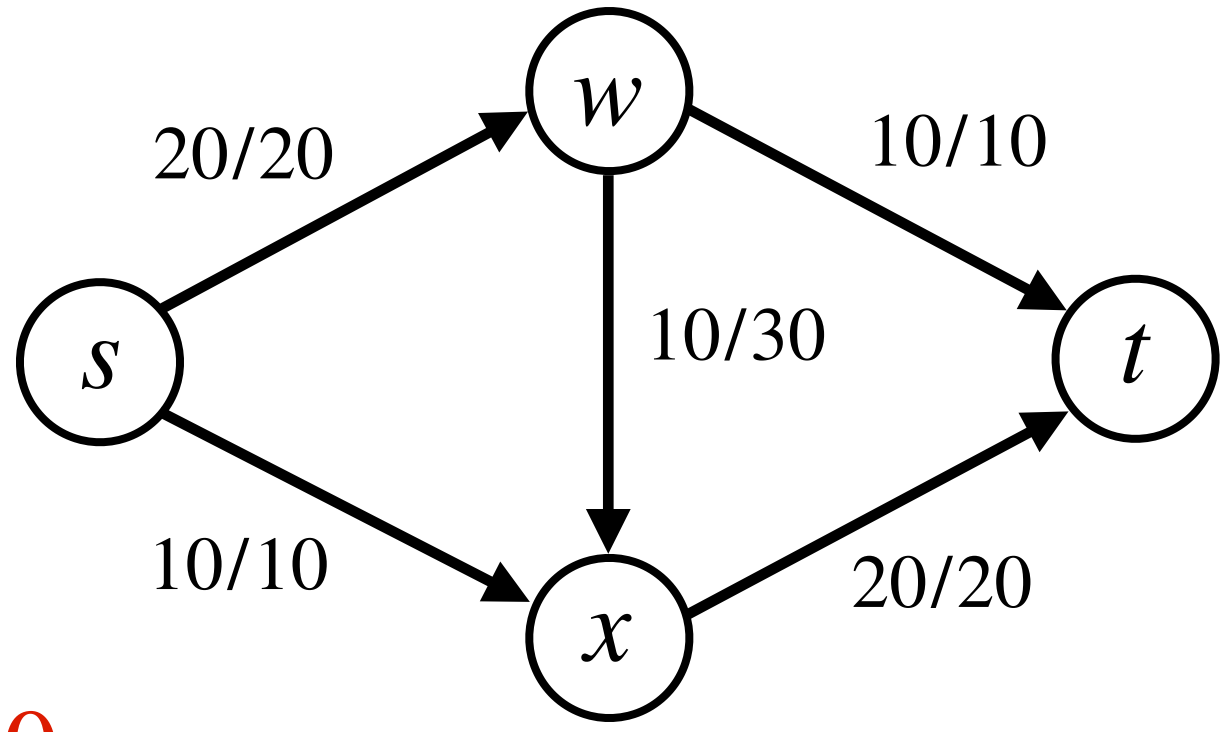

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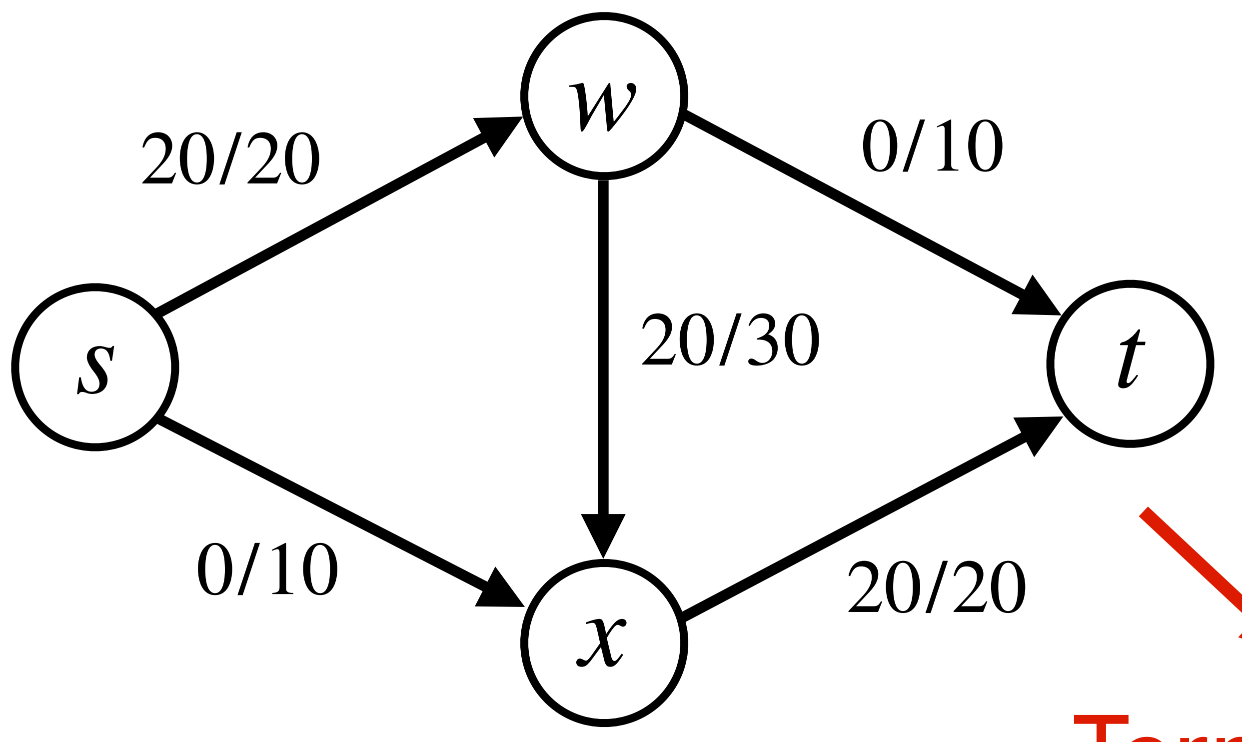

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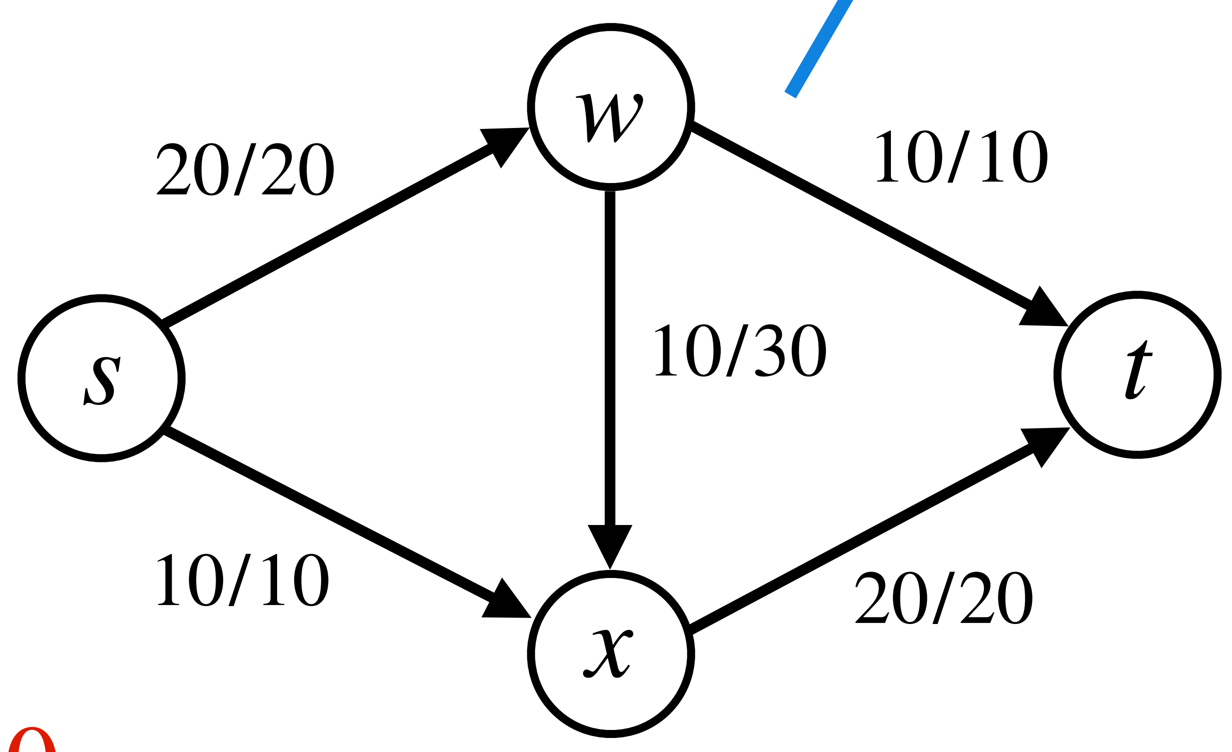
Max-flow is 30



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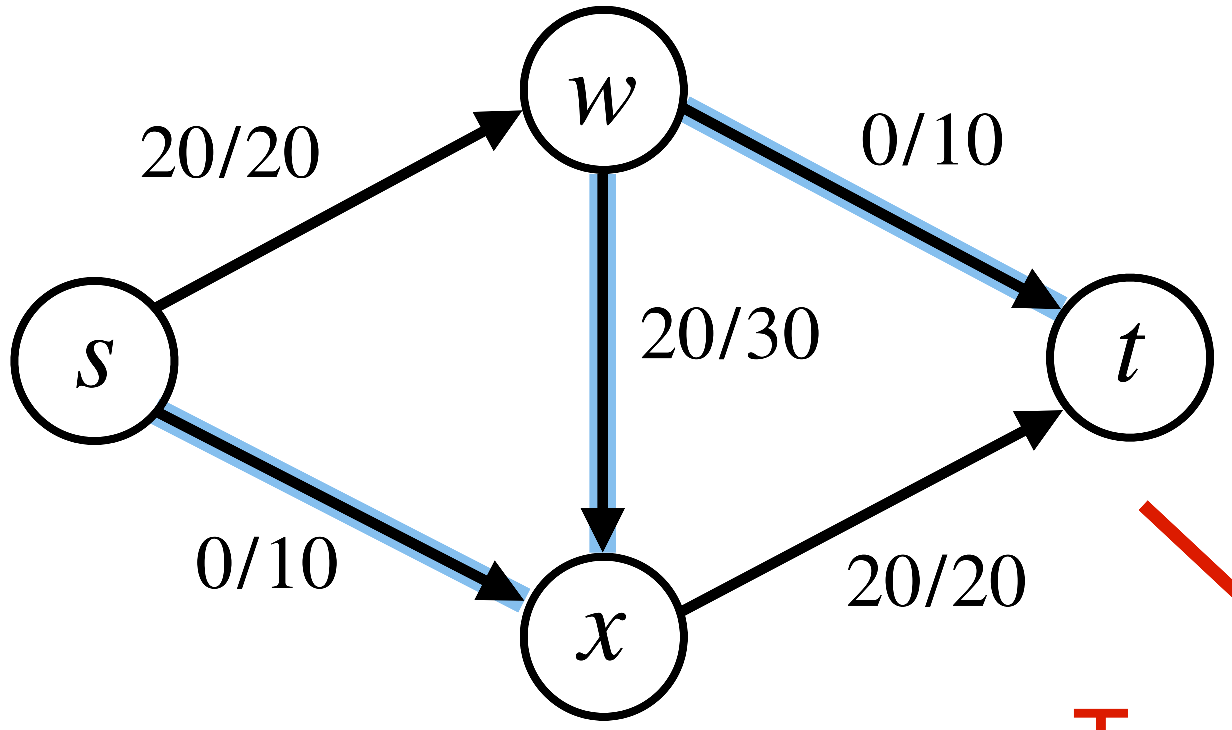

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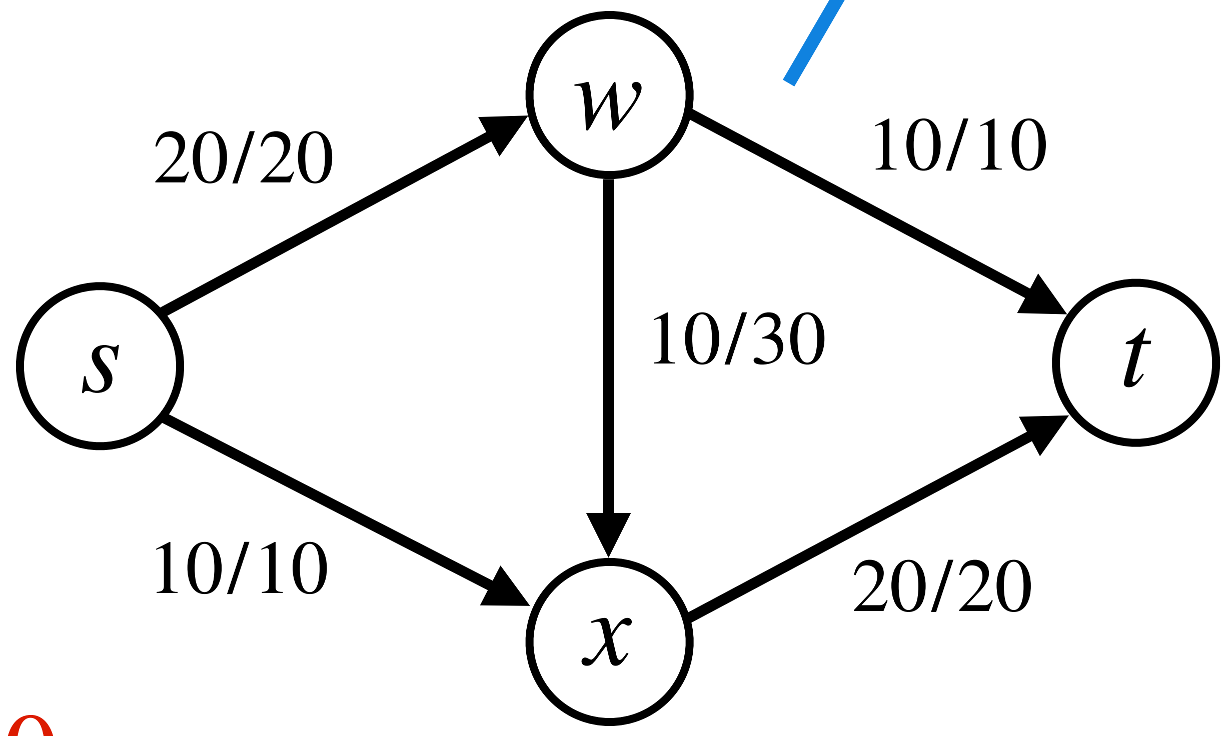

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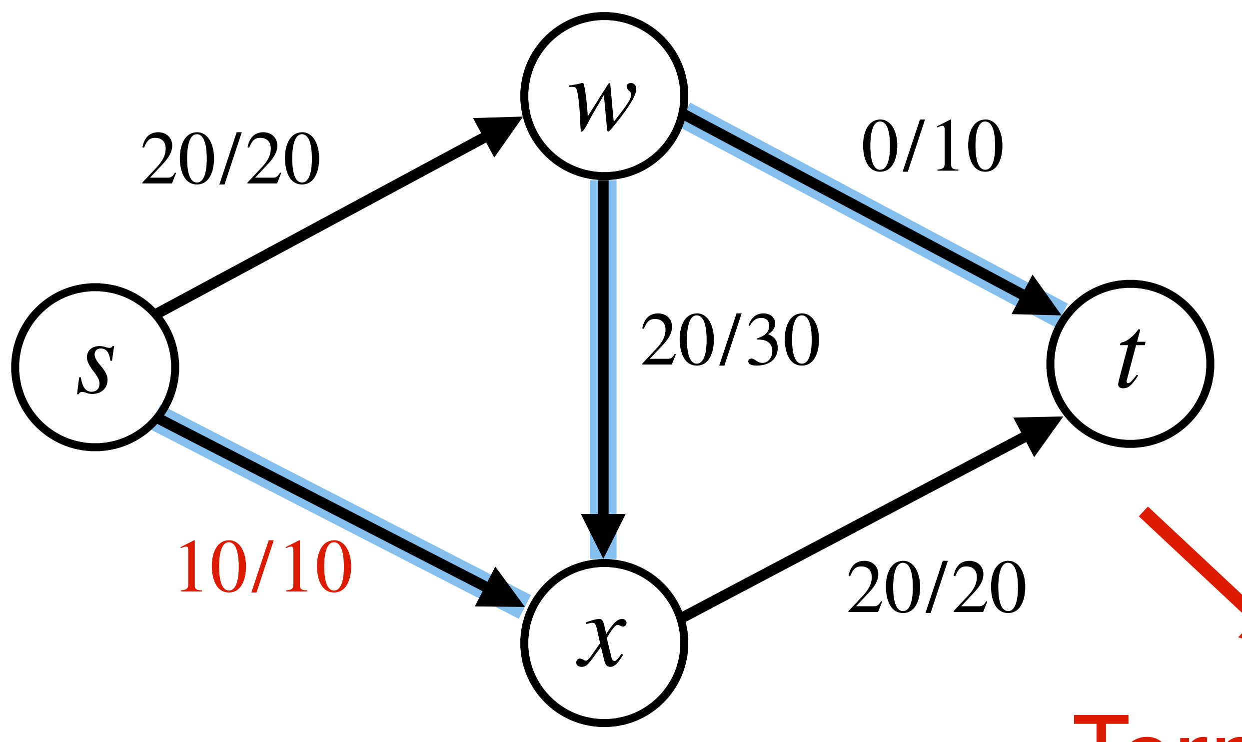

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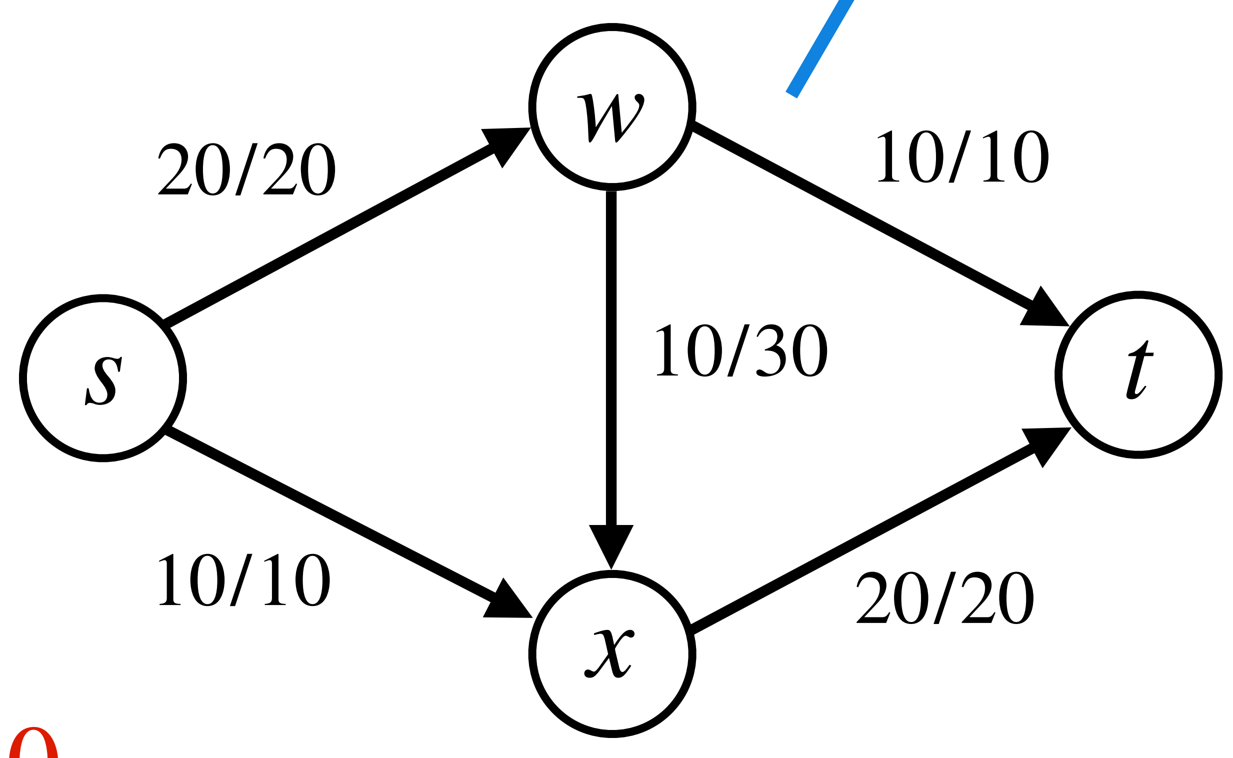

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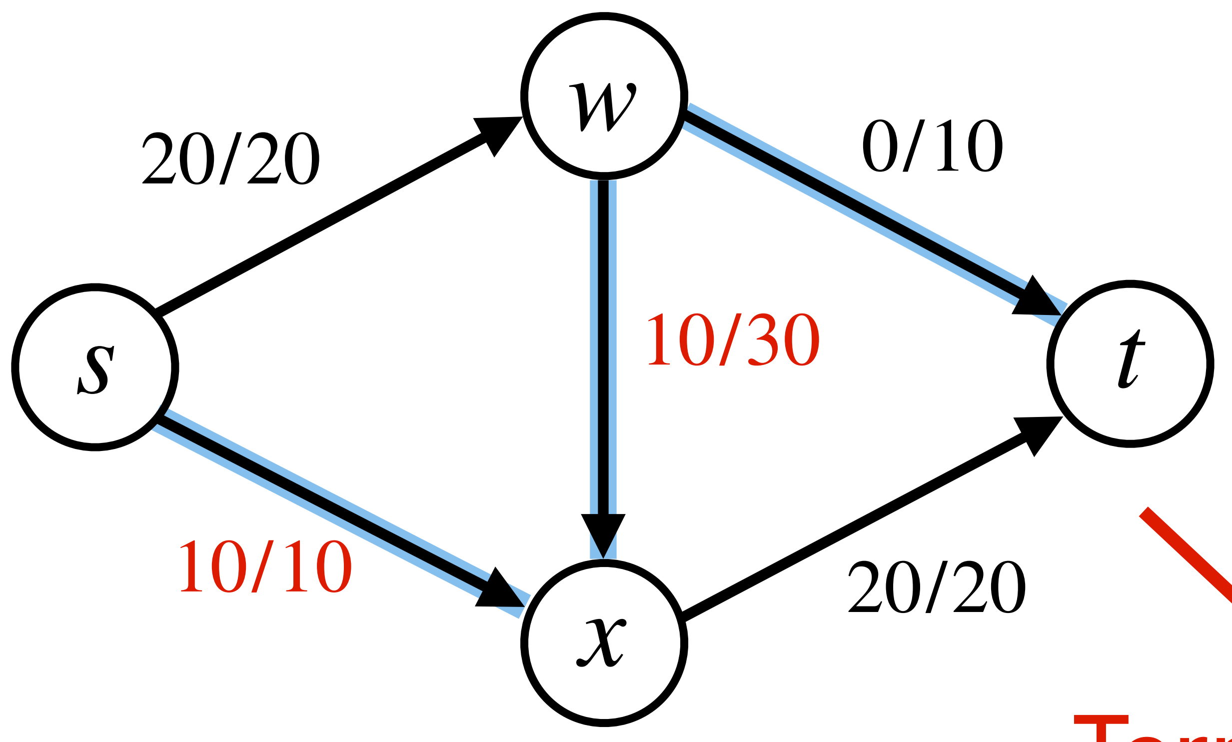

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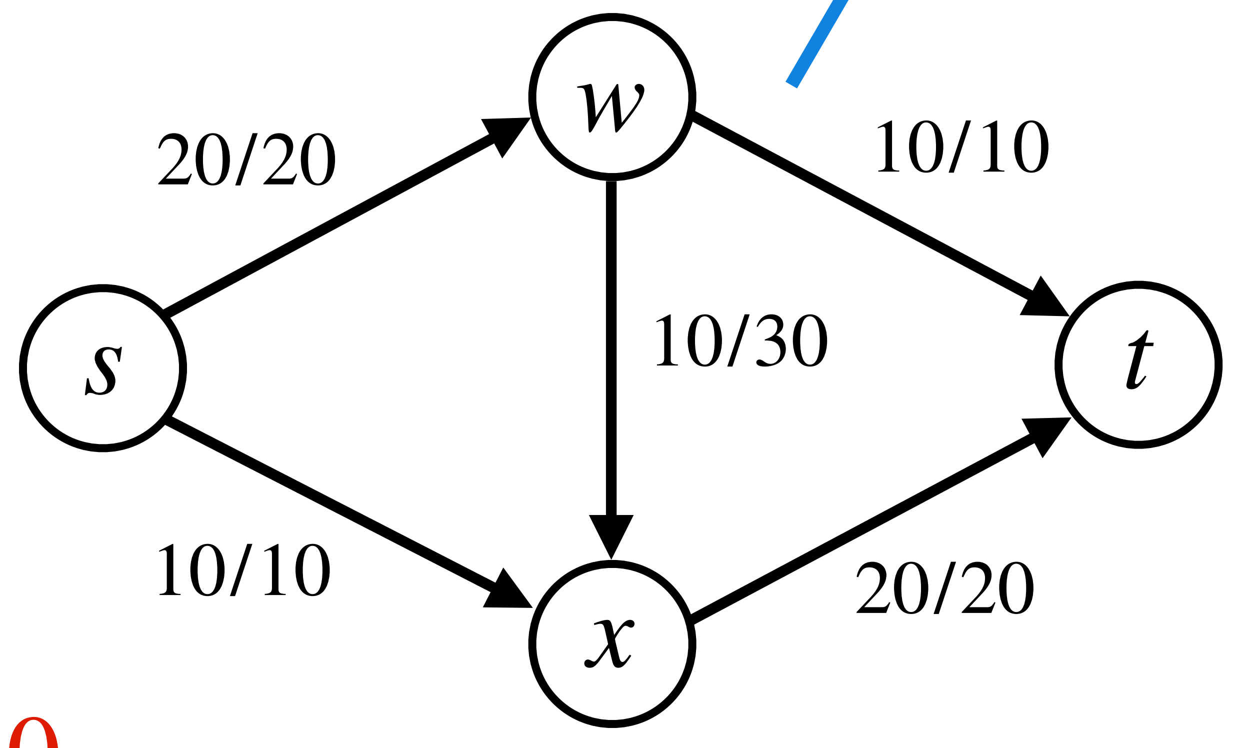

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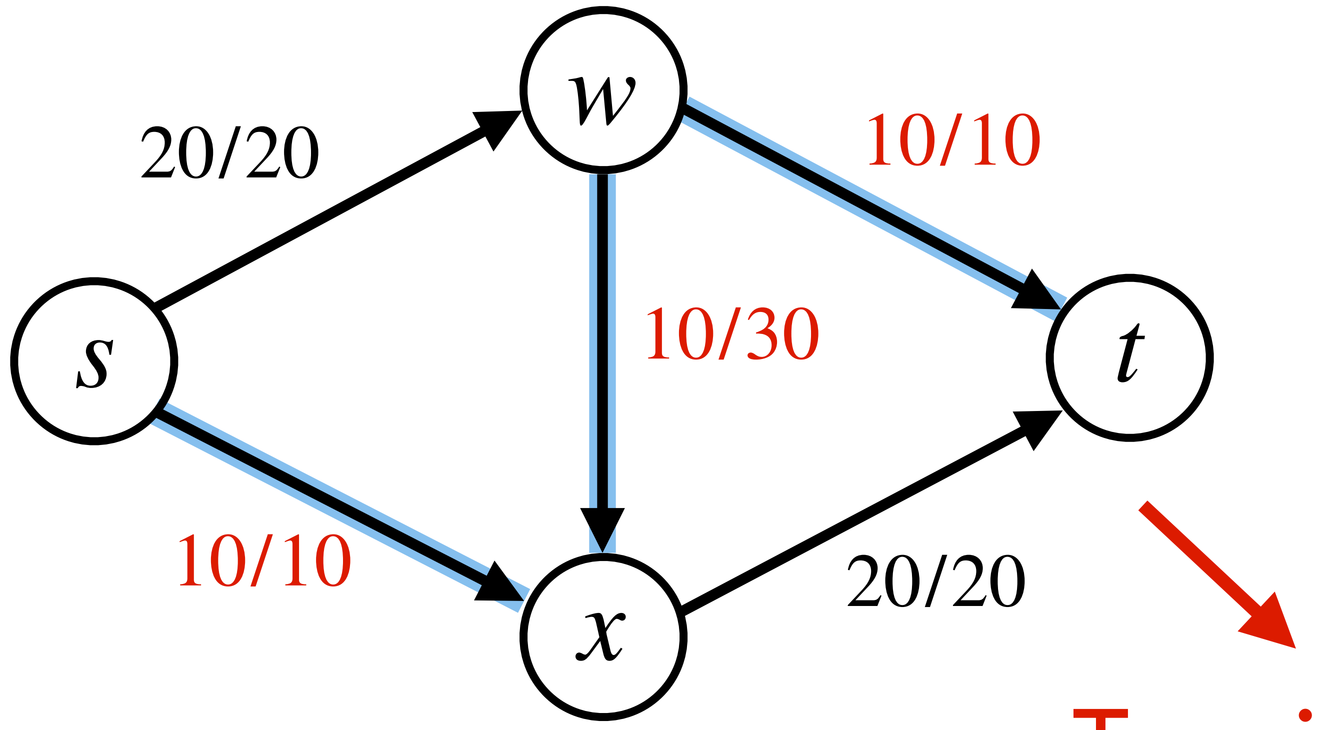

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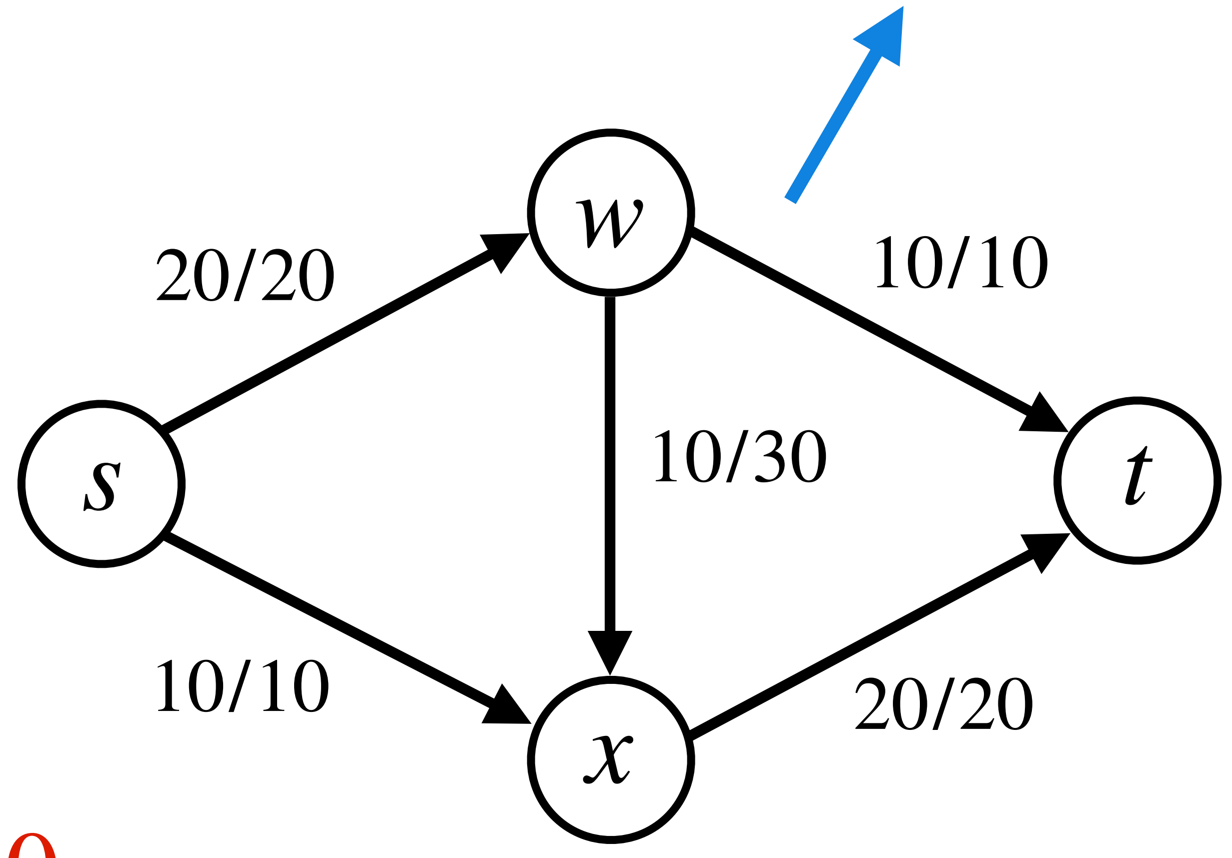

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Need to learn a new structure for that!

Residual Networks

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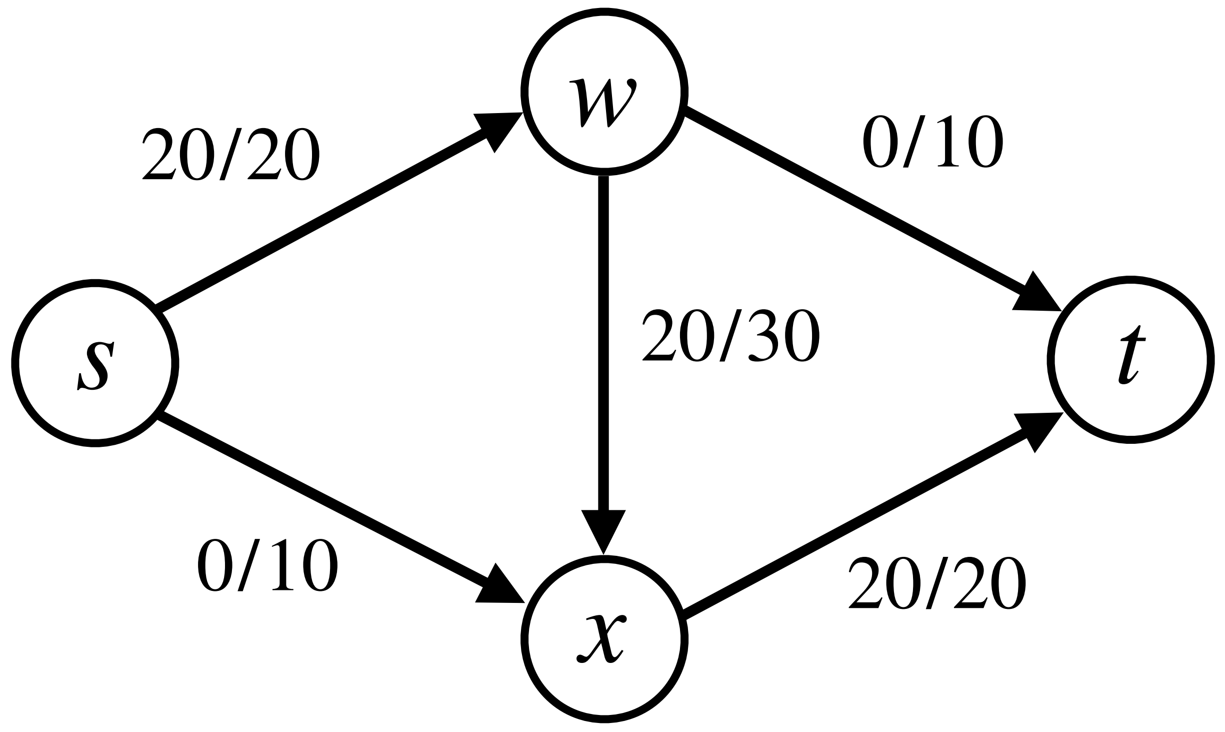

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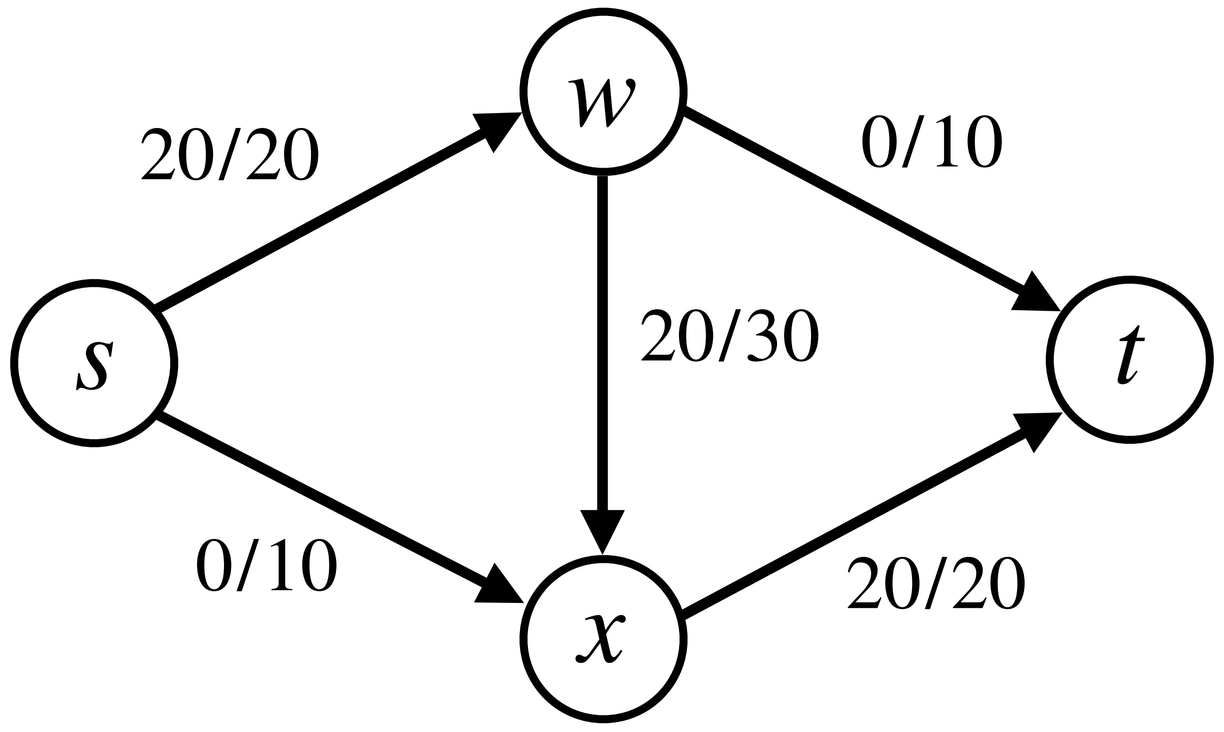

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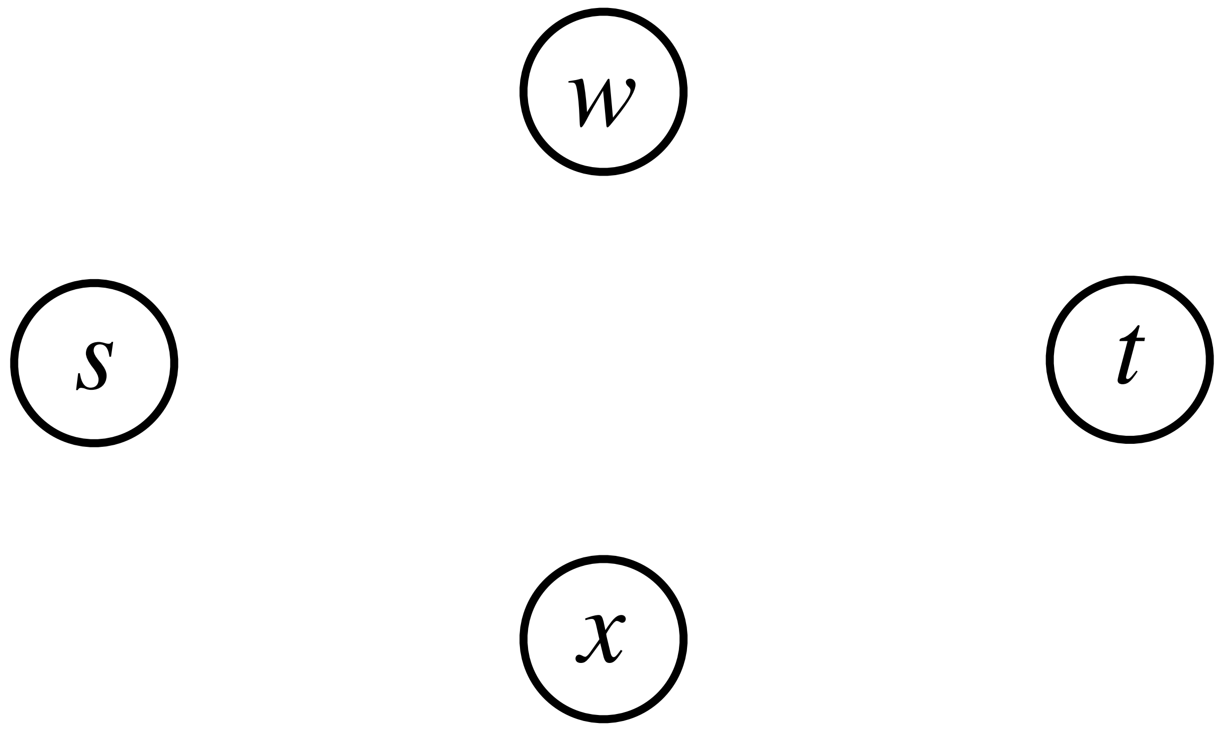

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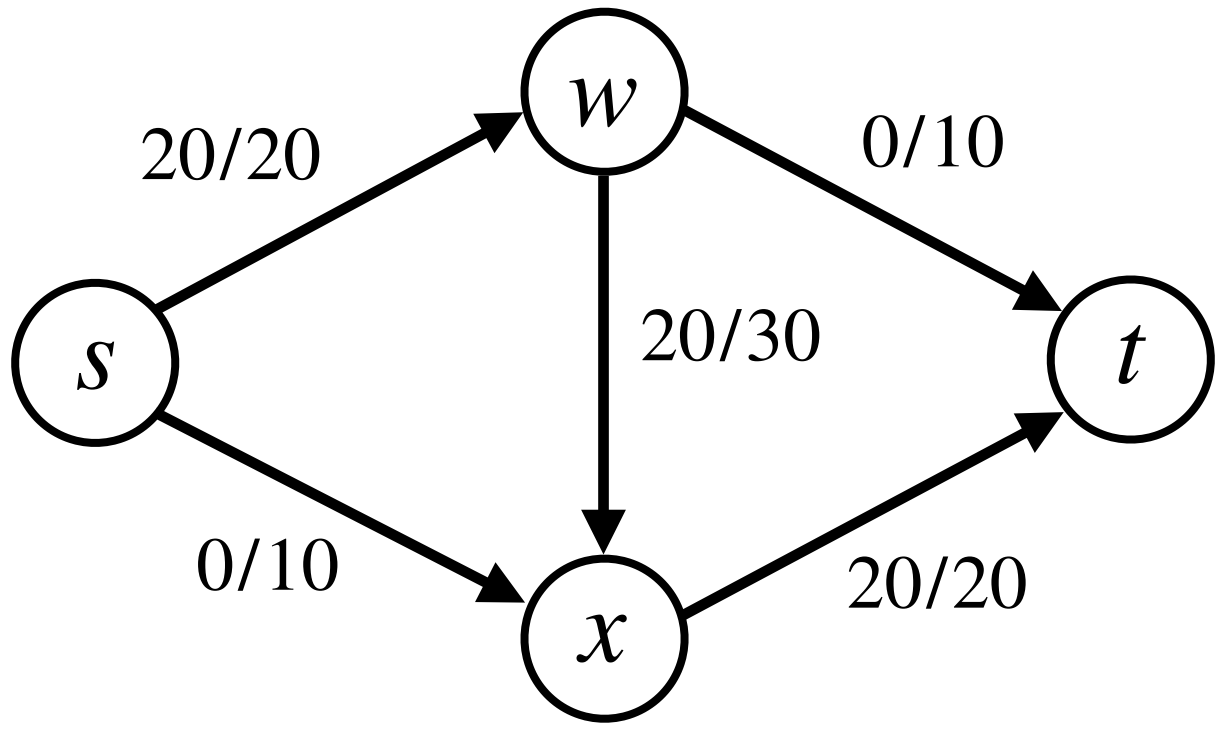

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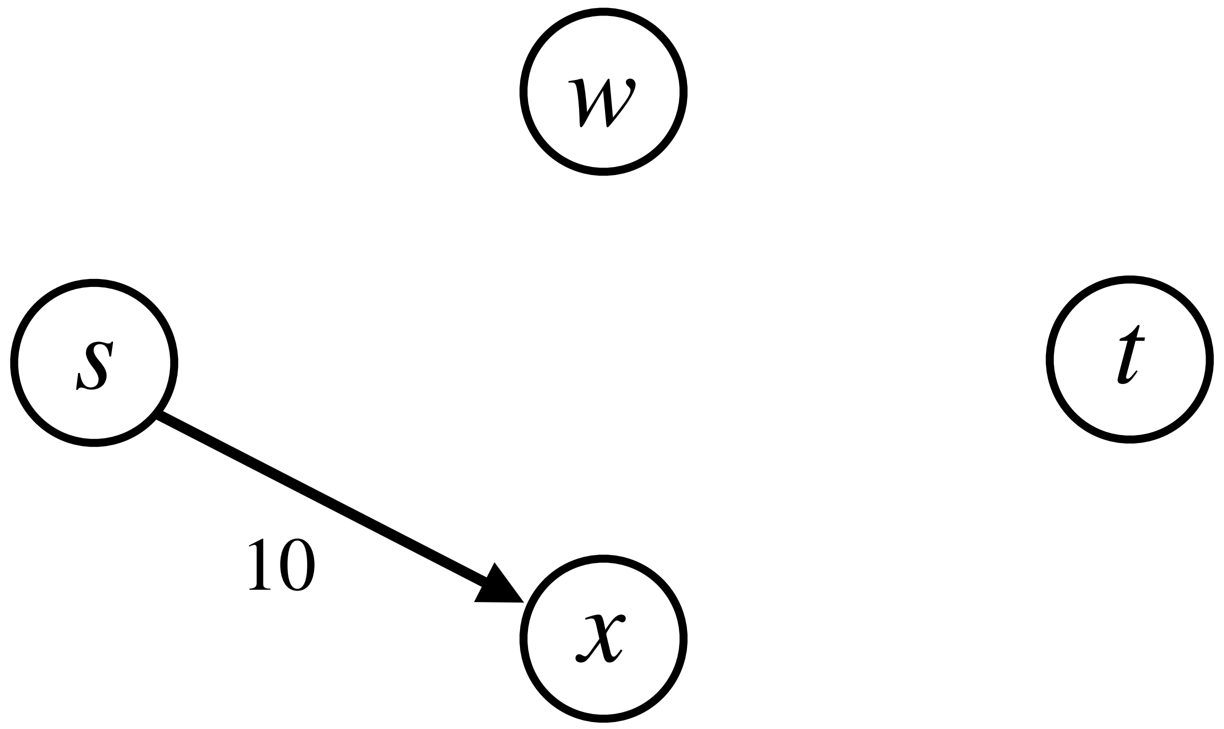

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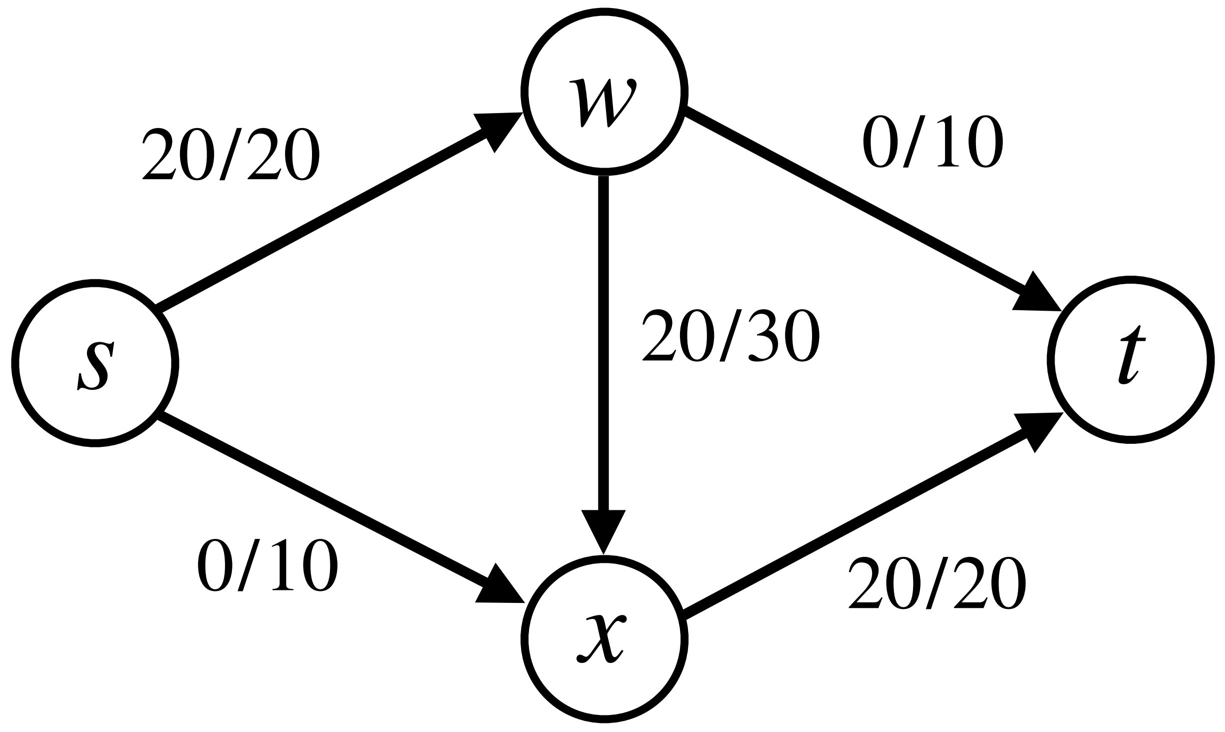

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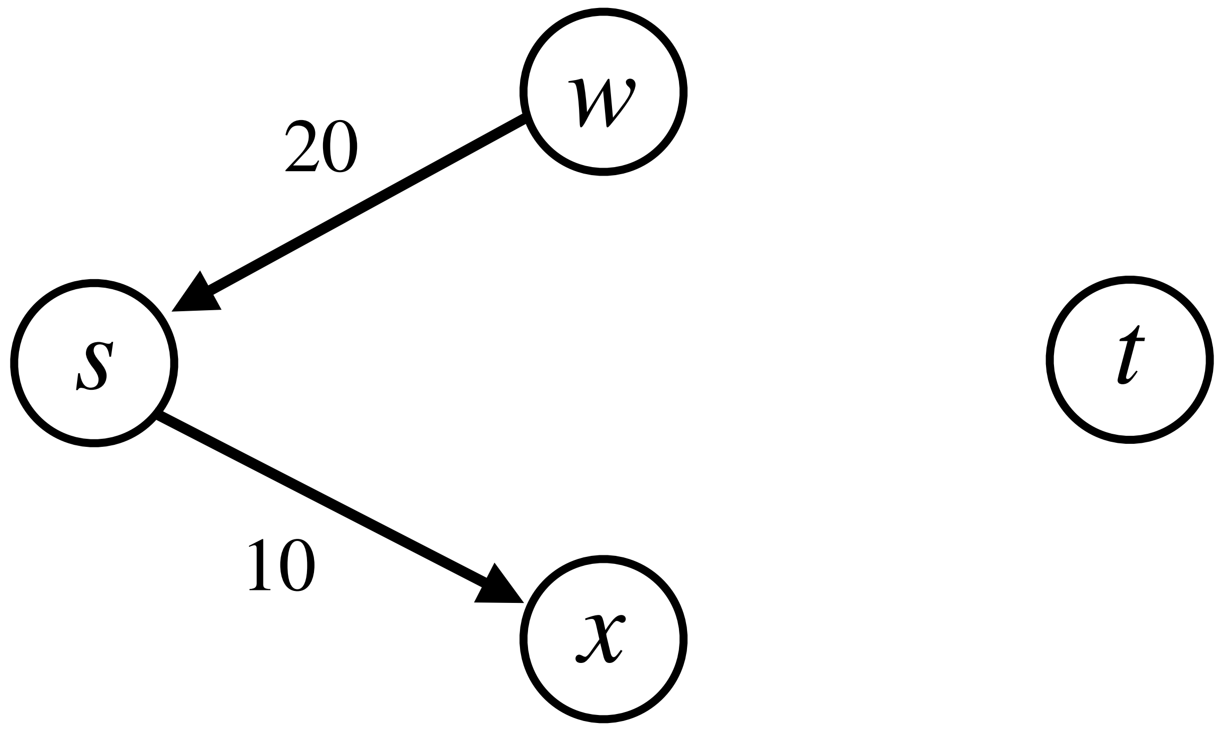

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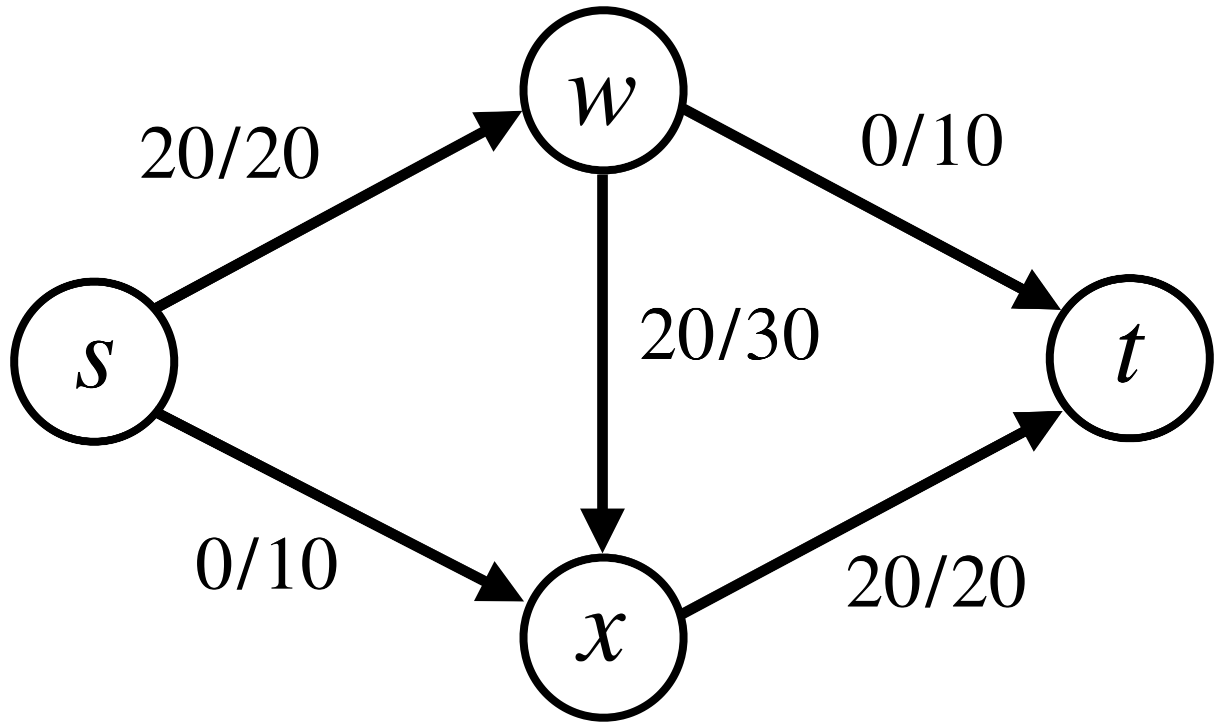

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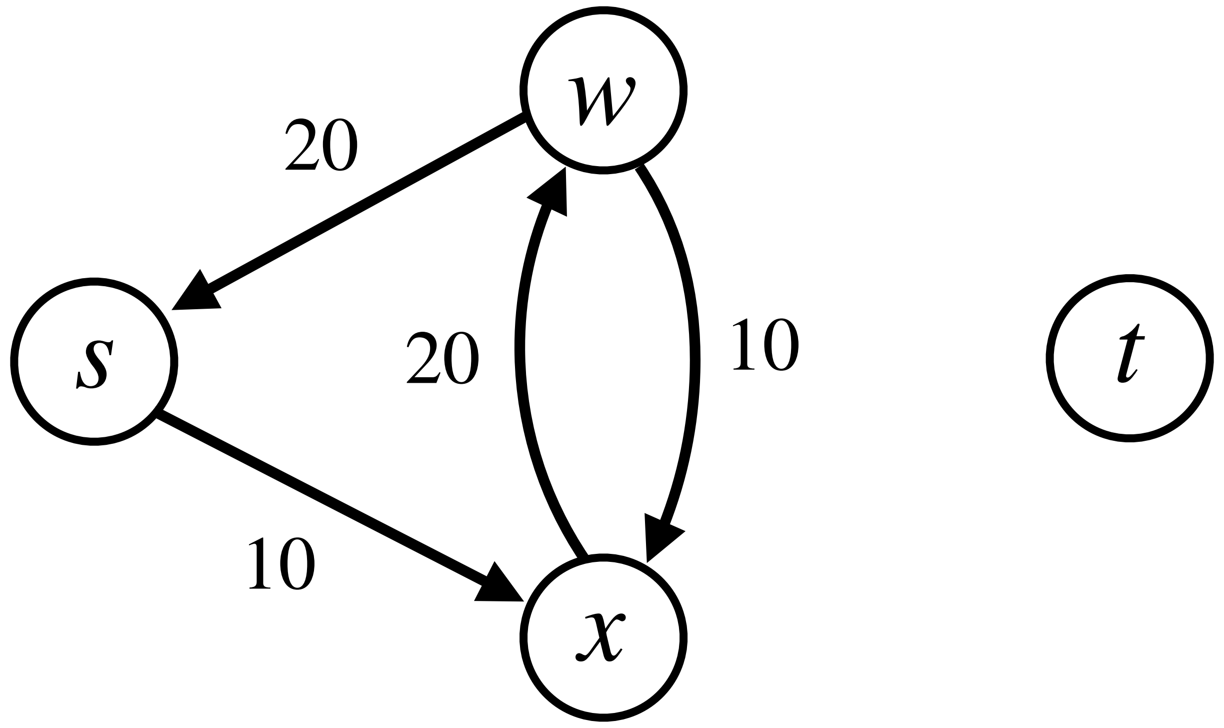

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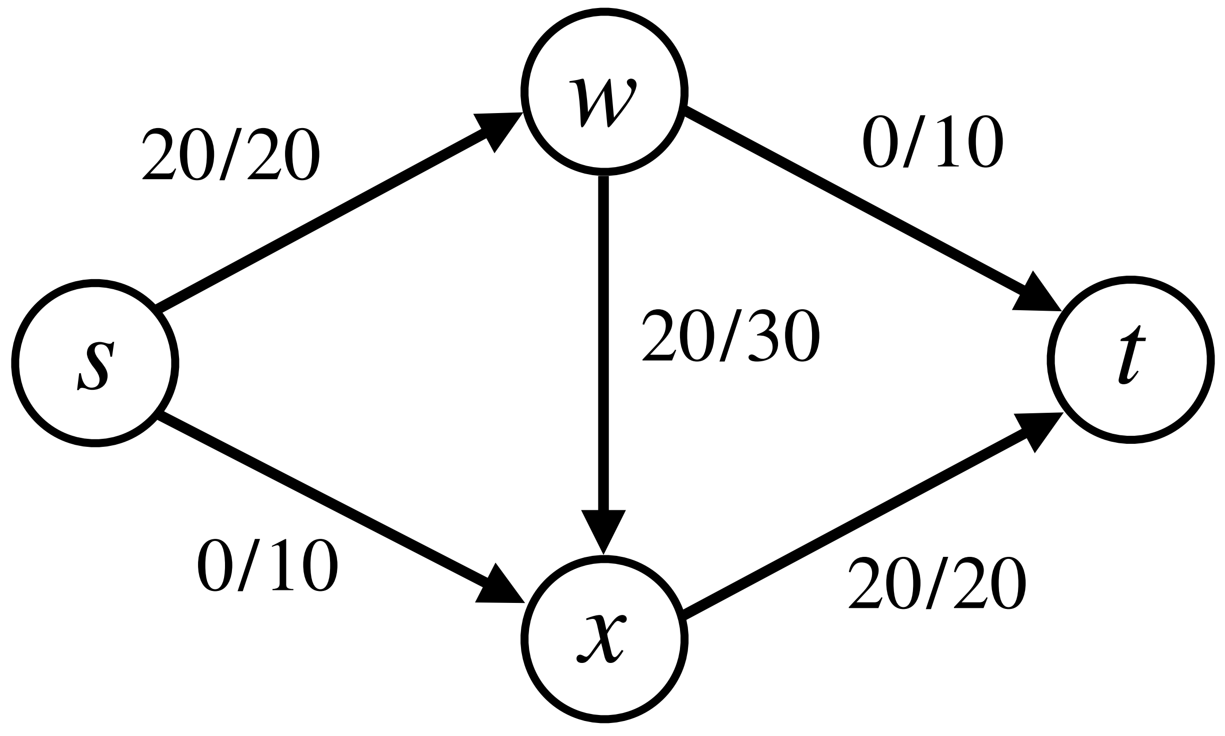

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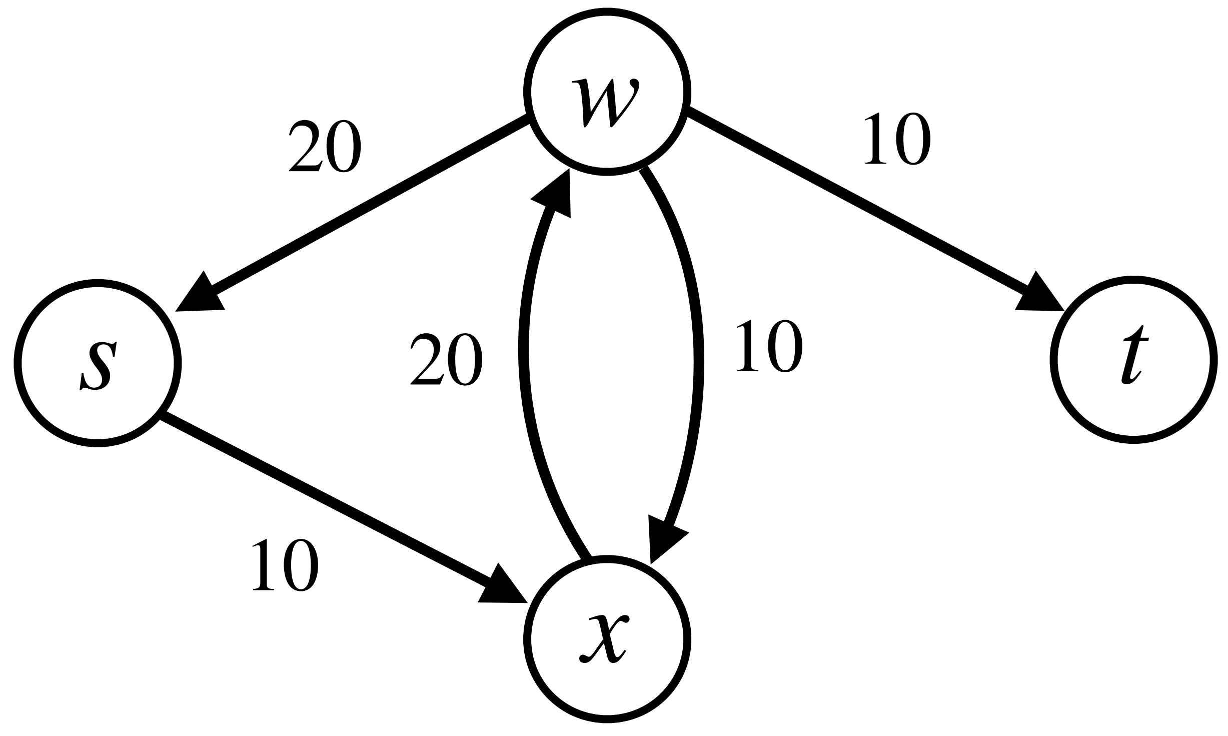

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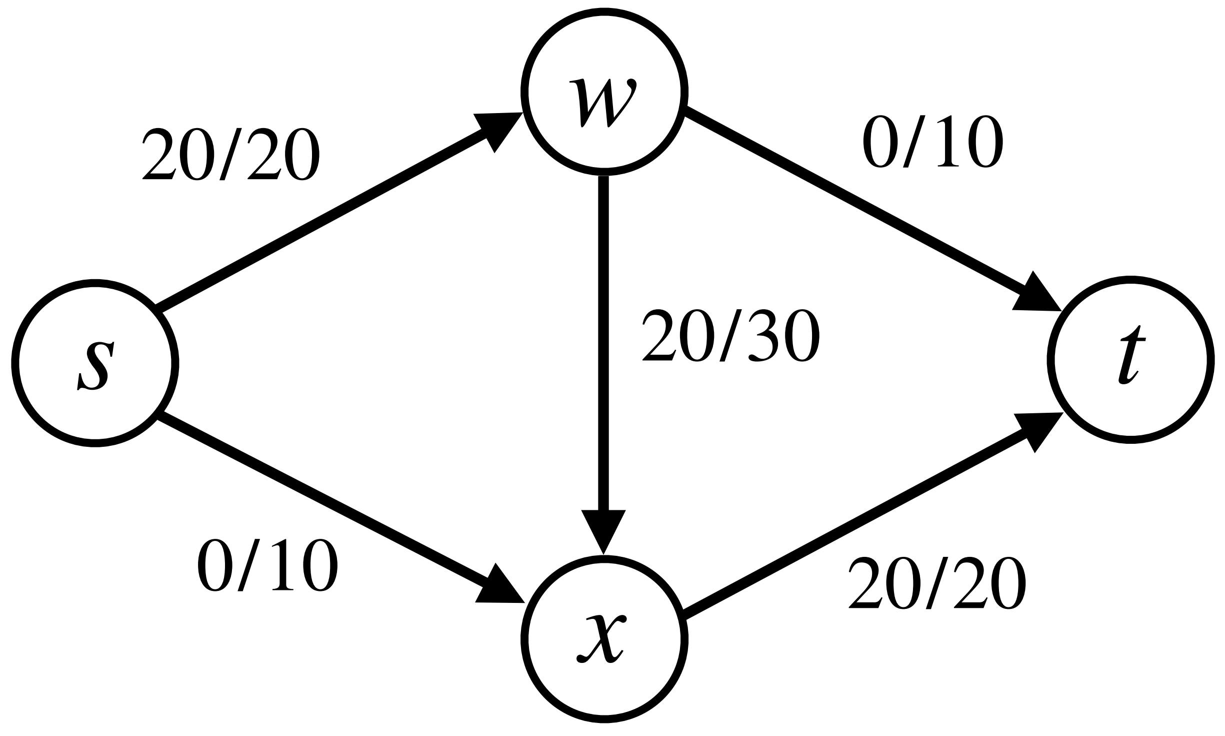

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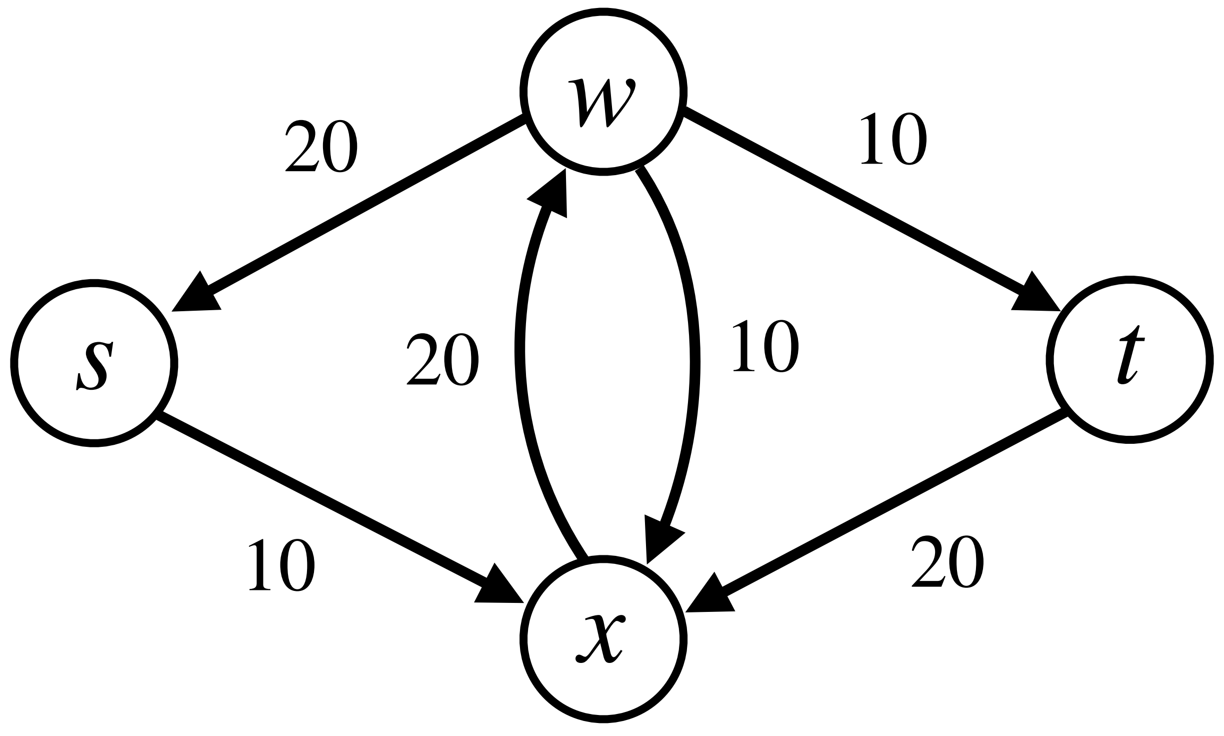

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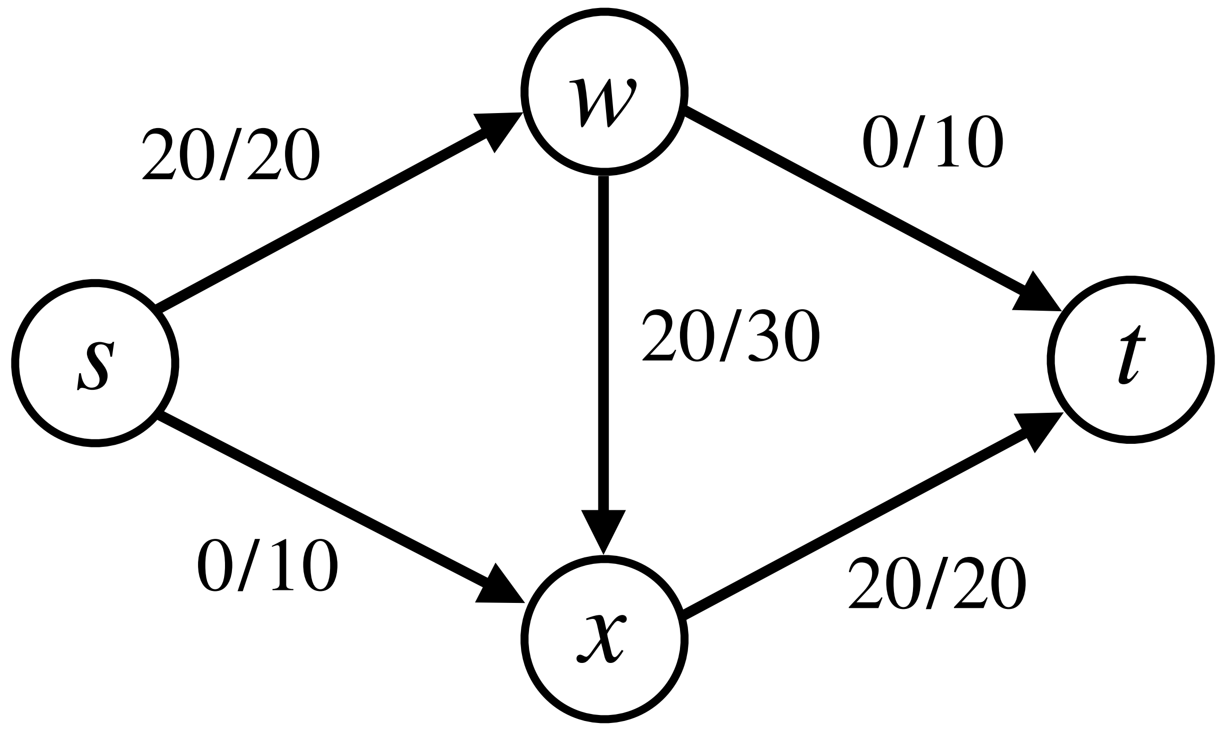
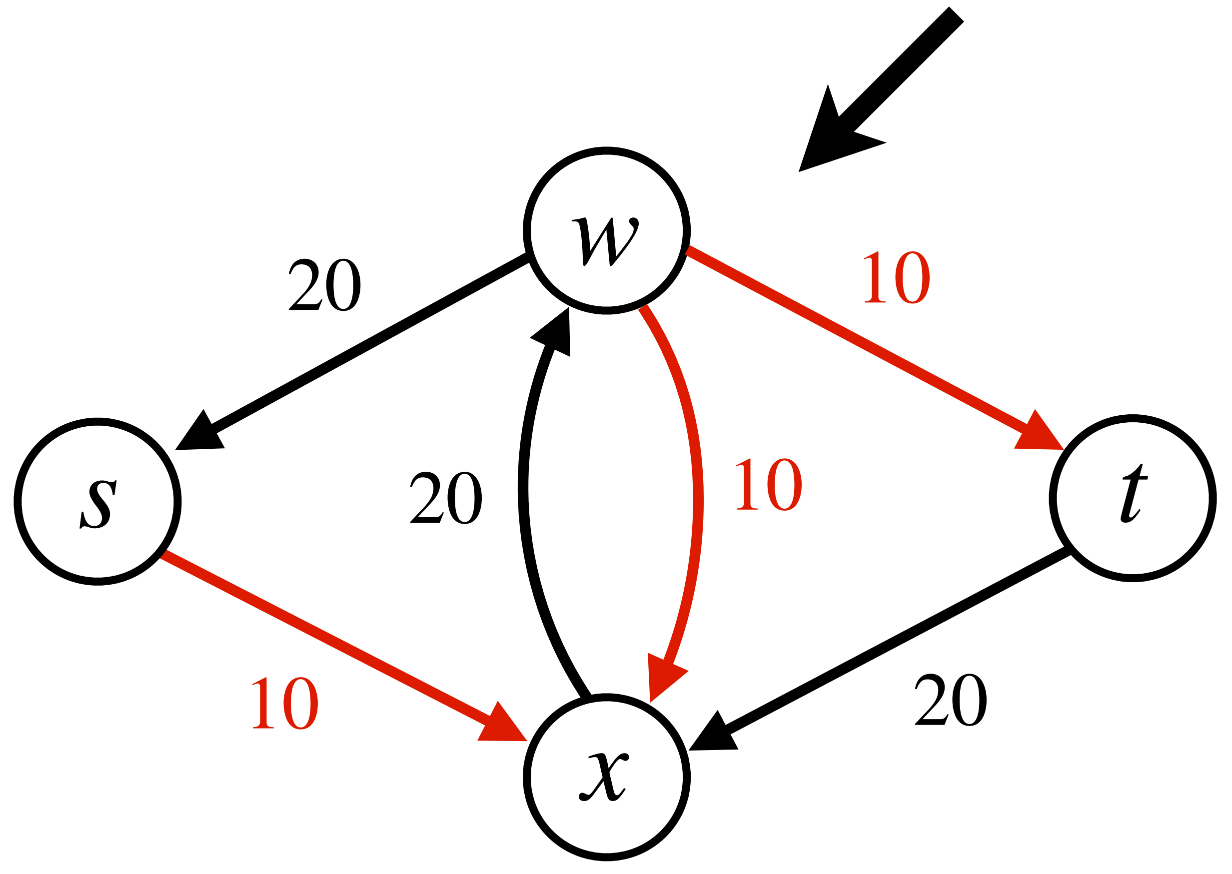


G

Forward edges to use the remaining capacity



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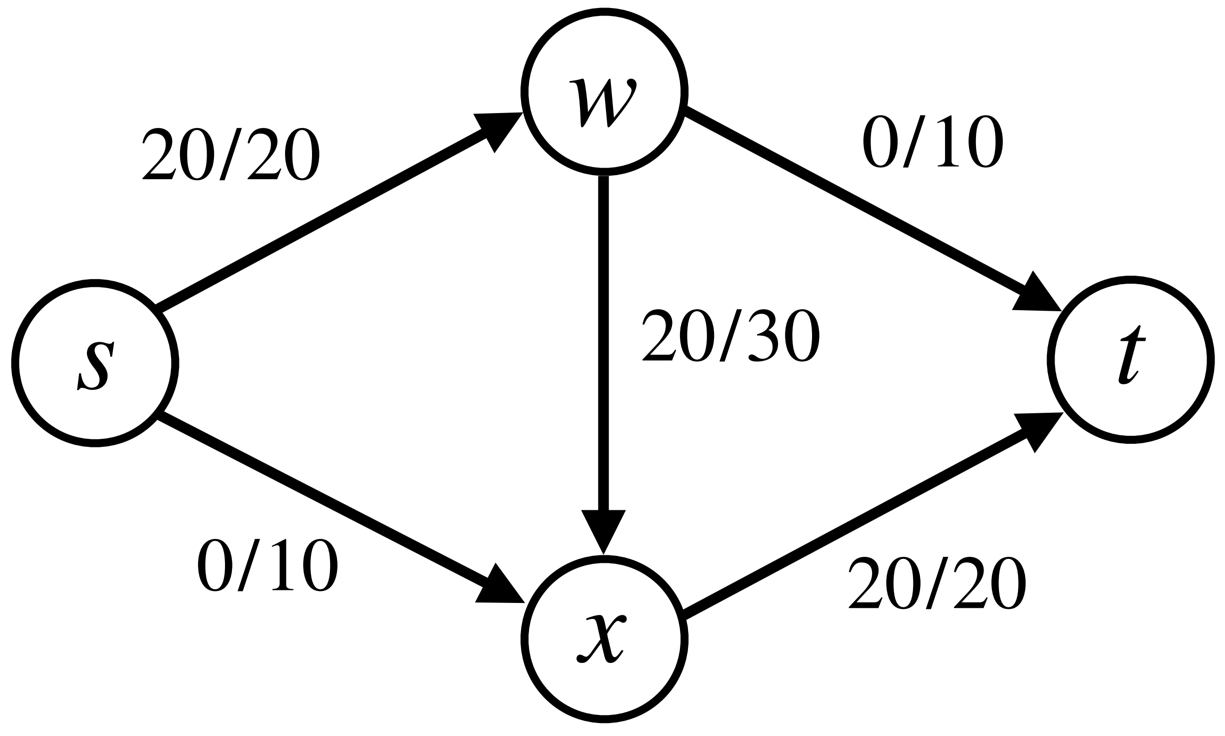
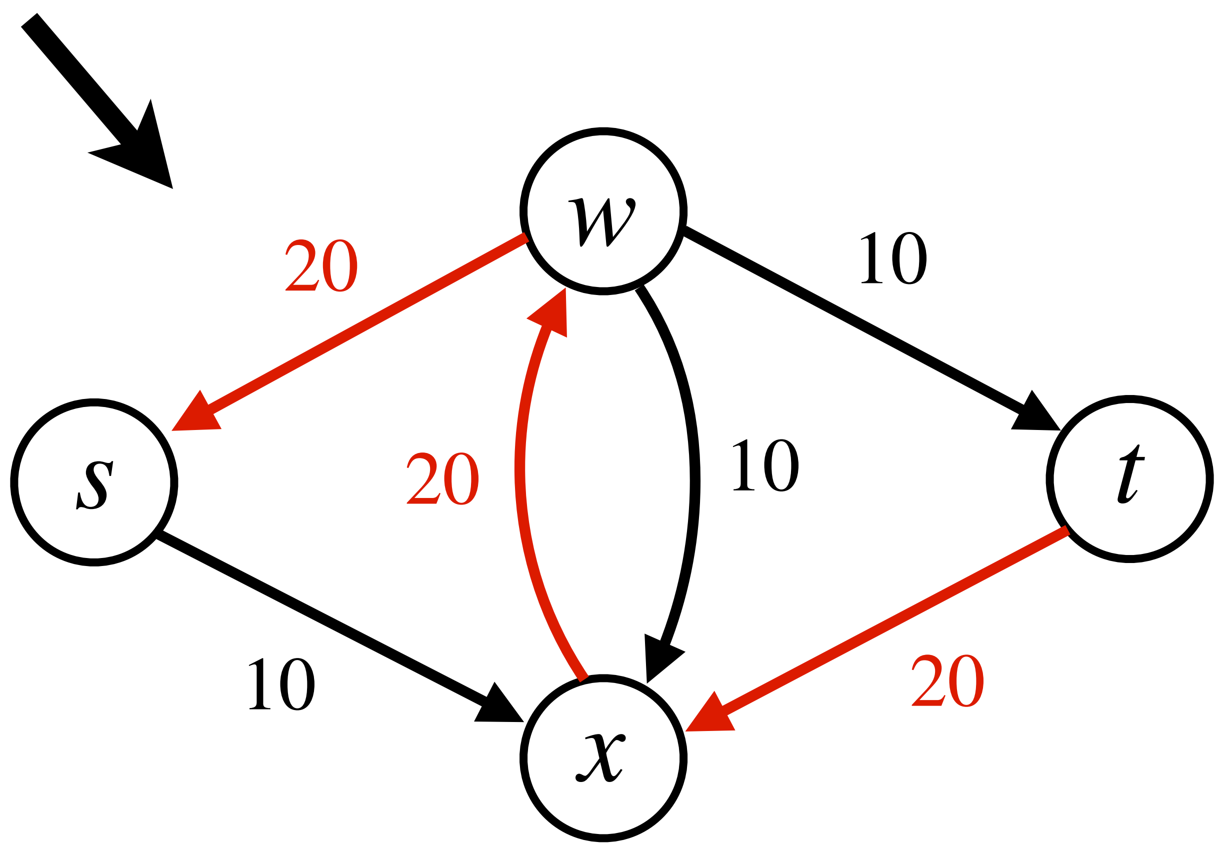


G

Backward edges to decrease the flow on some edges



G_f

Augmenting Flows via Residual Networks

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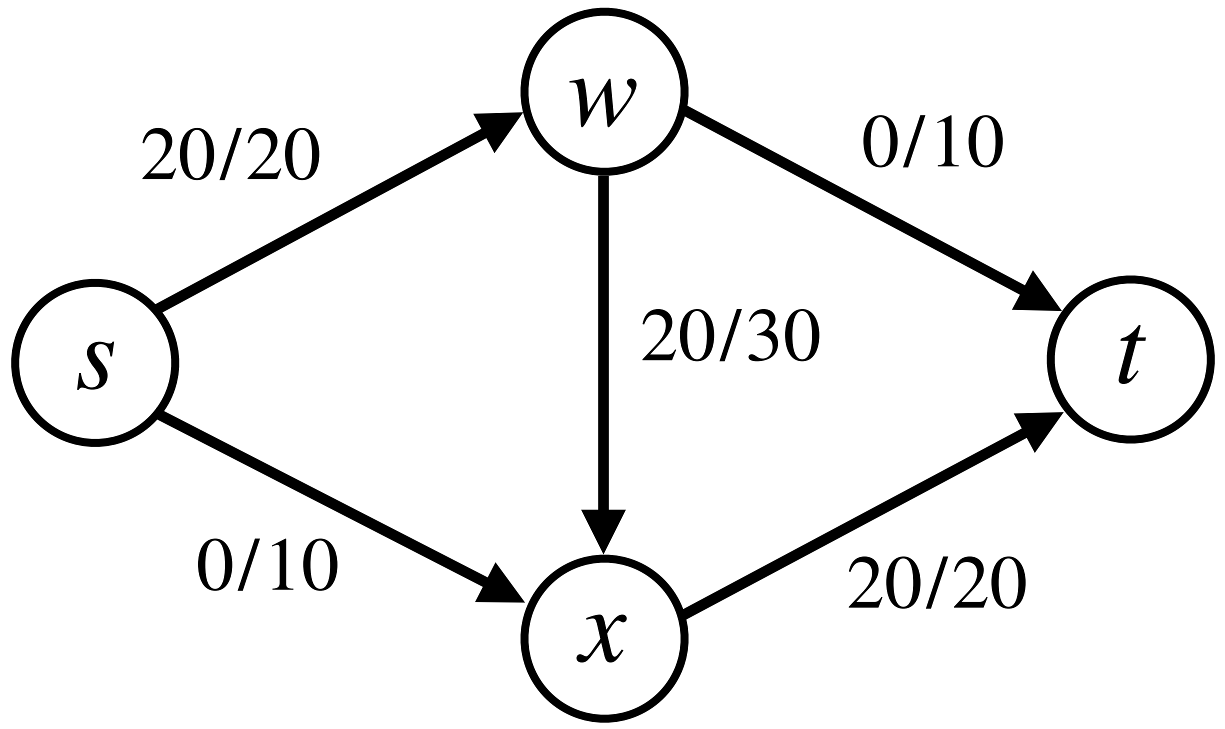

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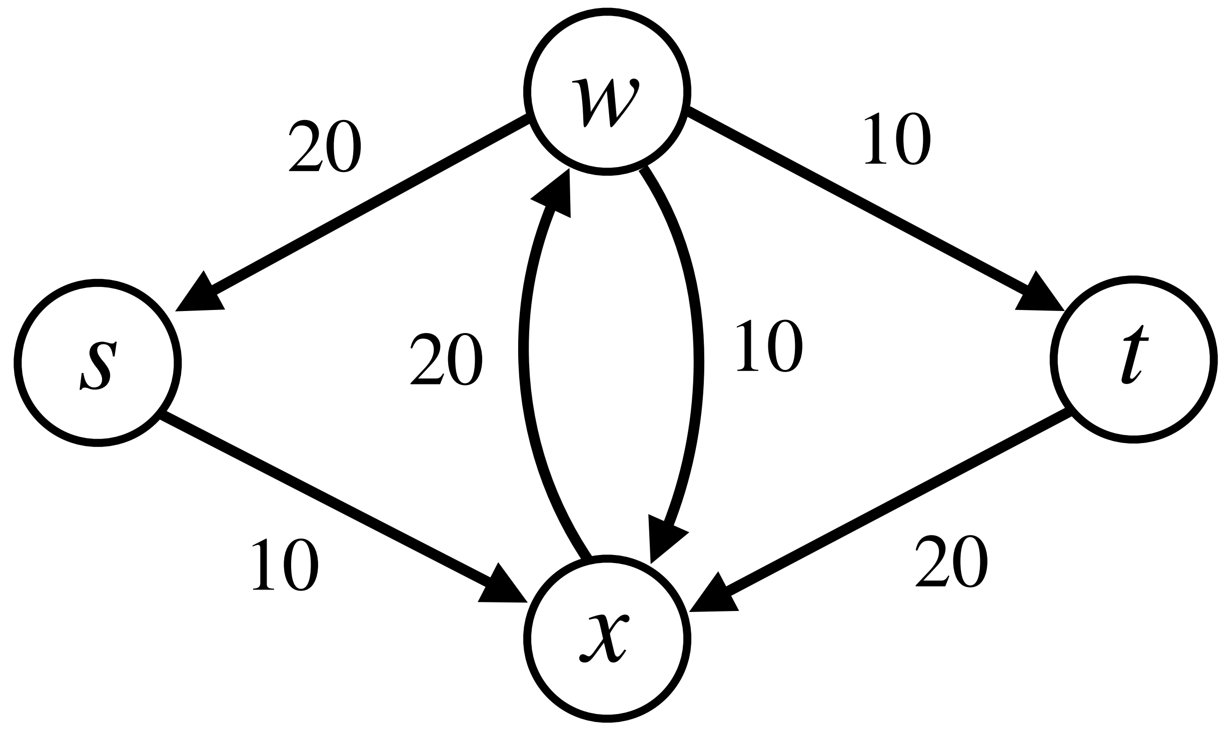

G_f

Augmenting Flows via Residual Networks

- Find $s \rightsquigarrow t$ path P in the **residual network** G_f and its bottleneck capacity δ.

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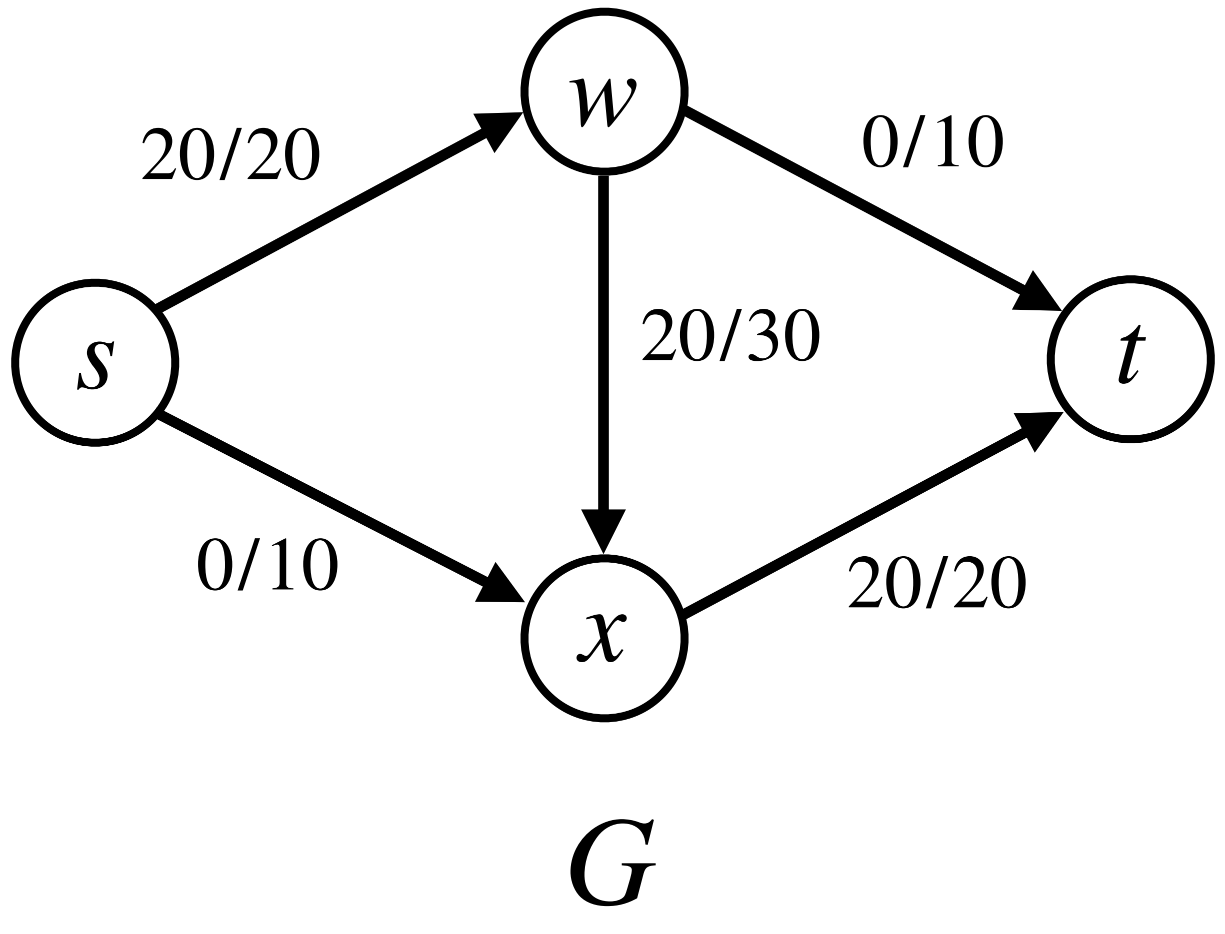

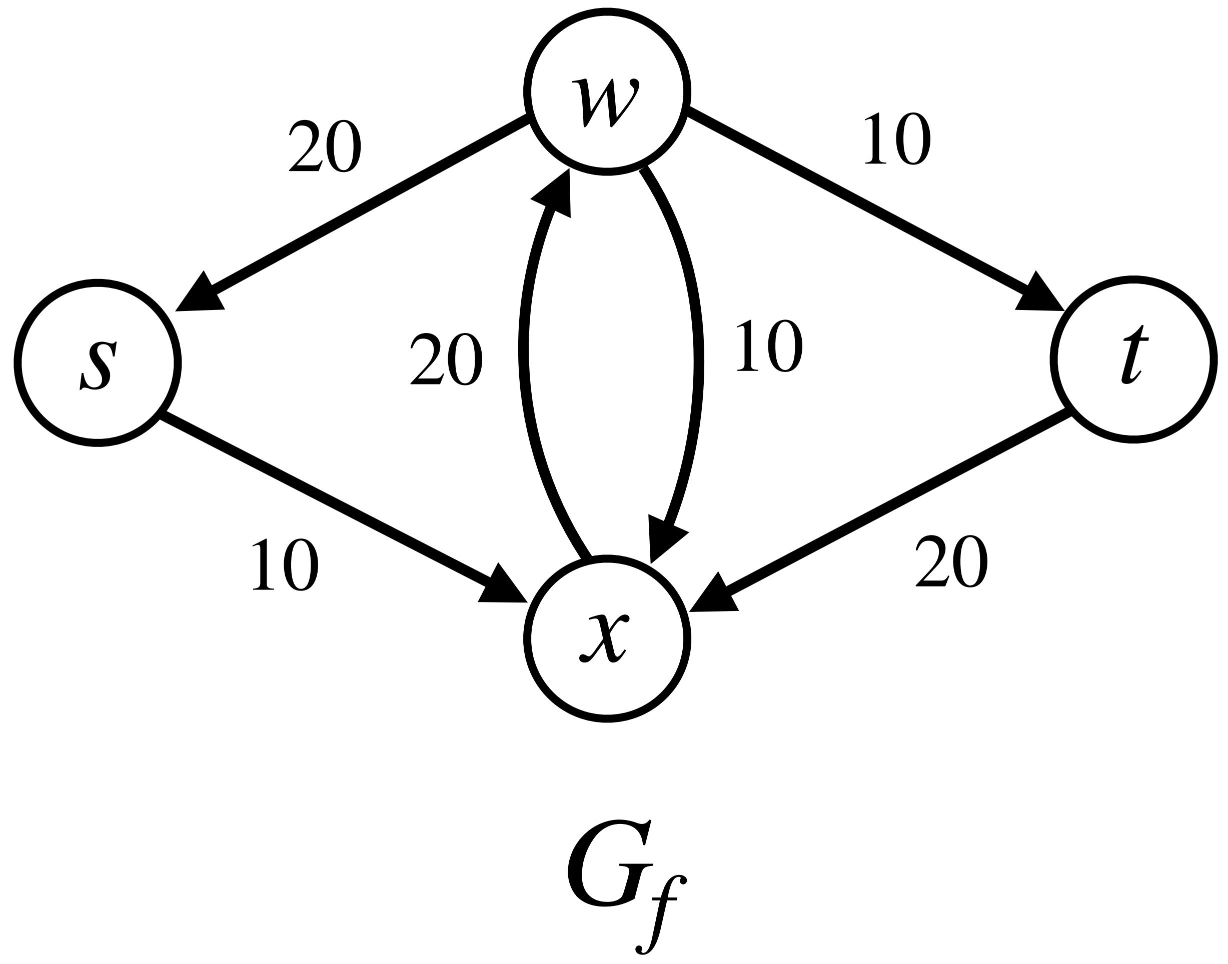

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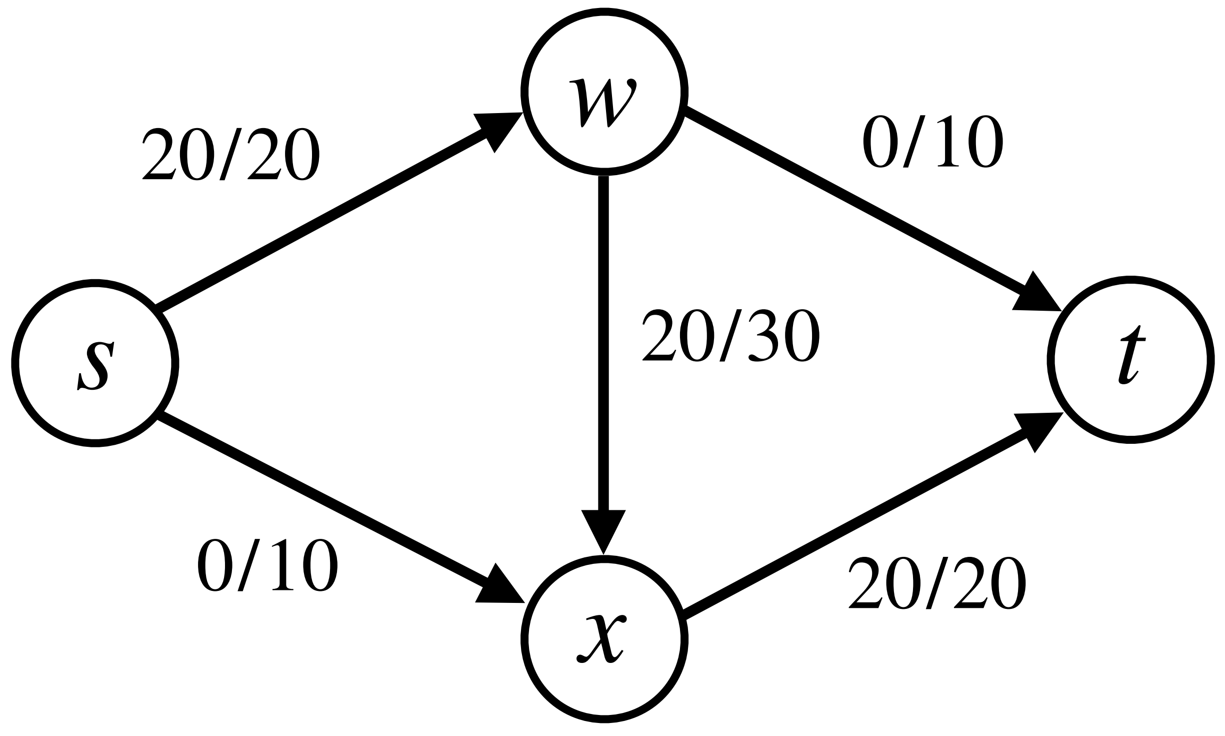
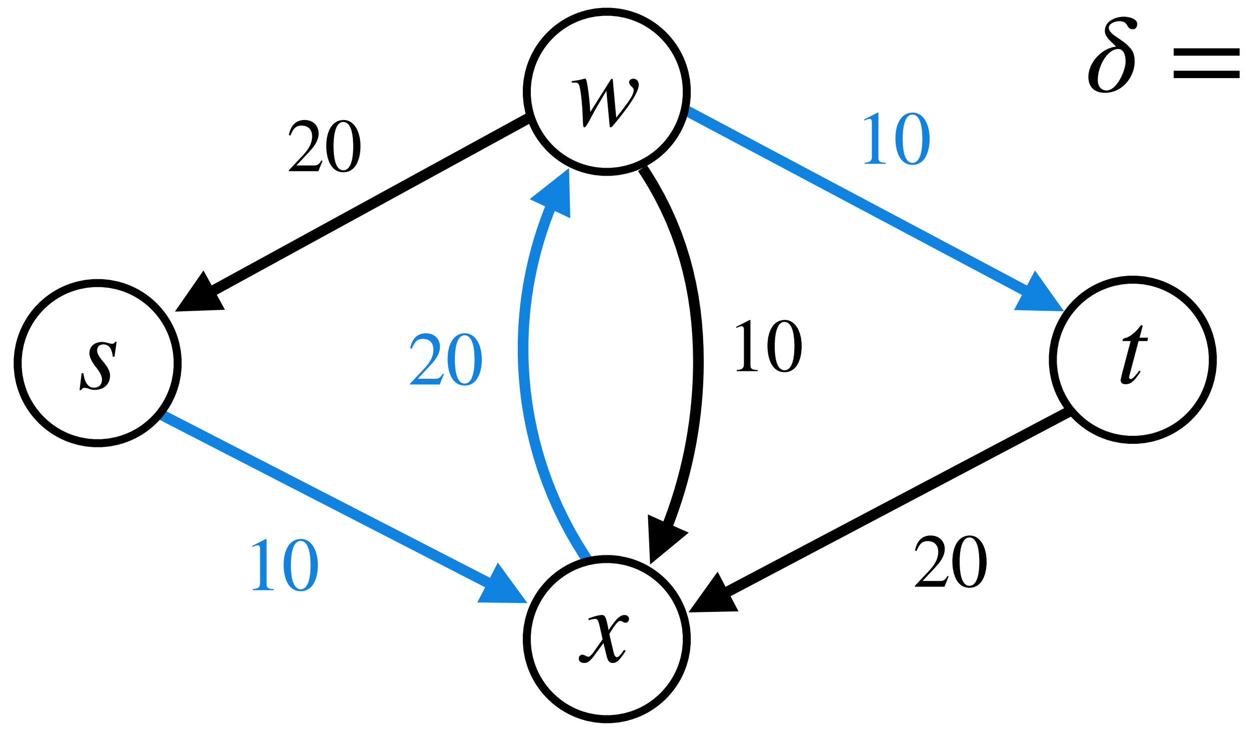


G

$\delta = 10$



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- For every $(u, v) \in P$:

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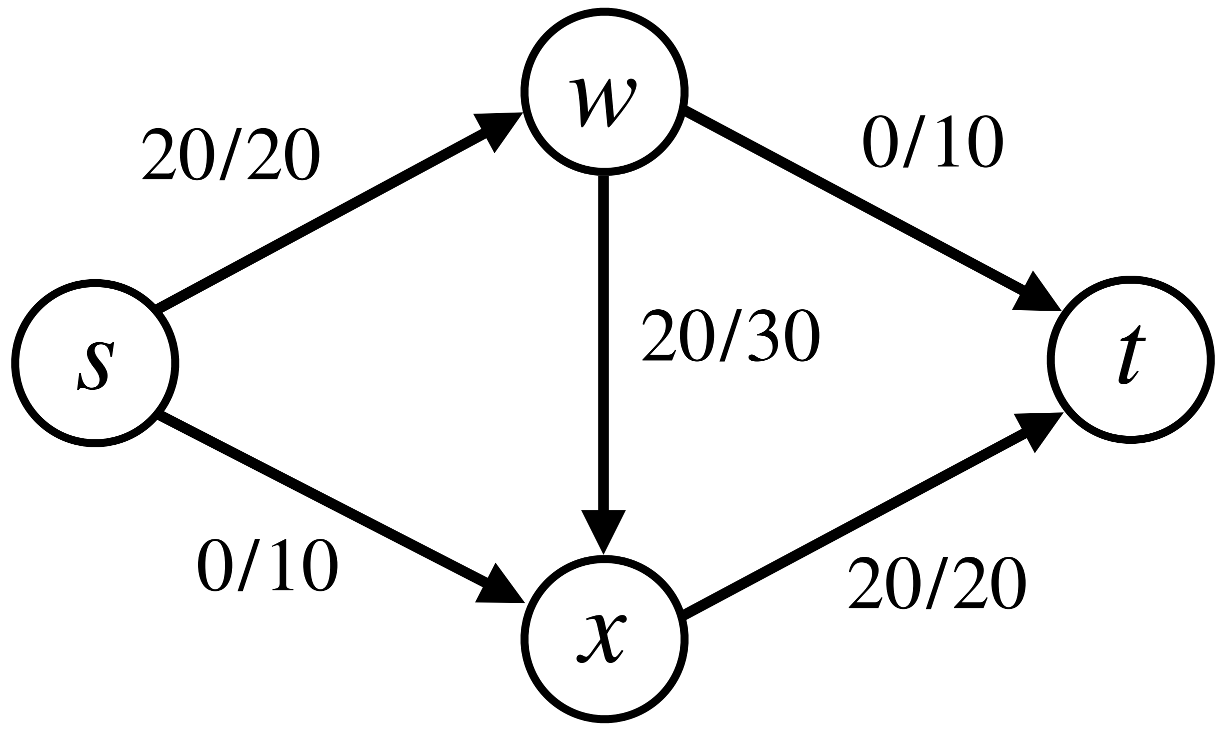
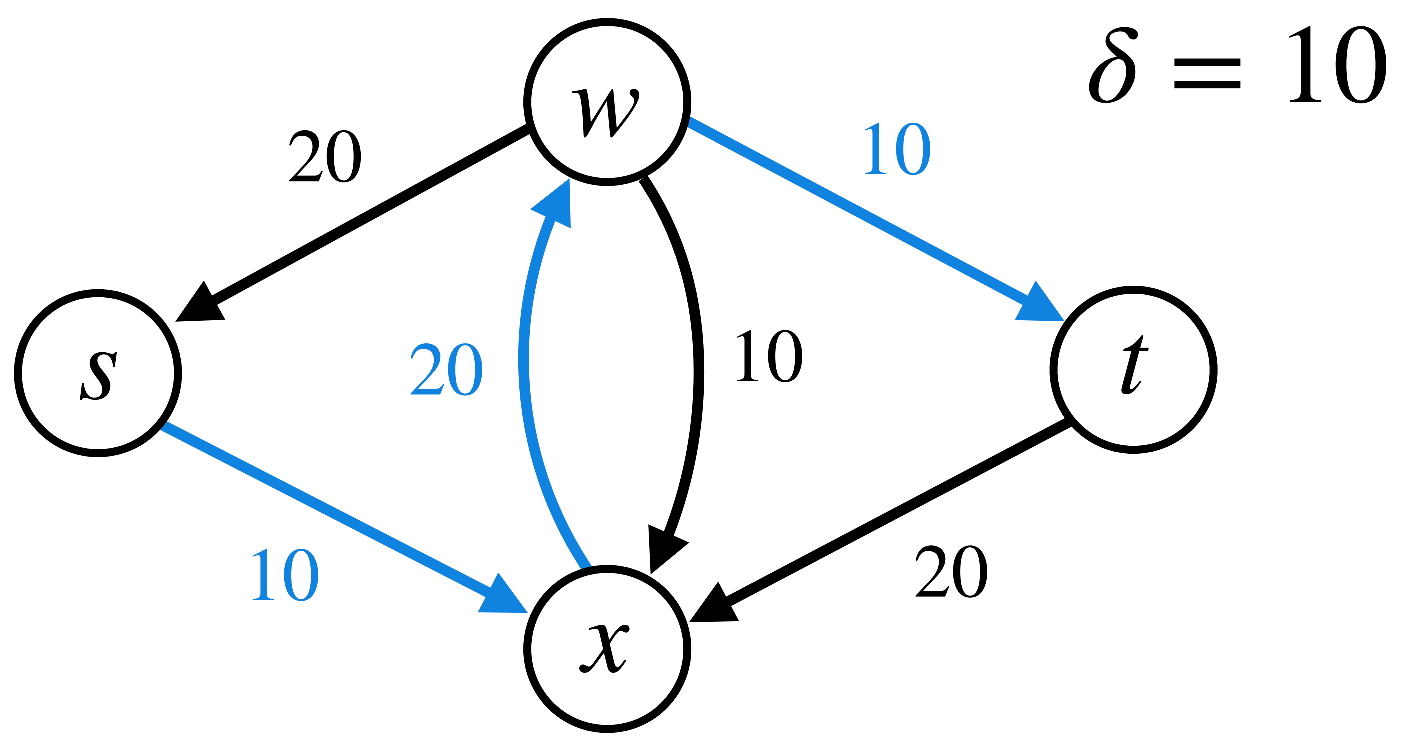


G



G_f

Augmenting Flows via Residual Networks

- Find $s \rightsquigarrow t$ path P in the **residual network** G_f and its bottleneck capacity δ.
- For every $(u, v) \in P$:
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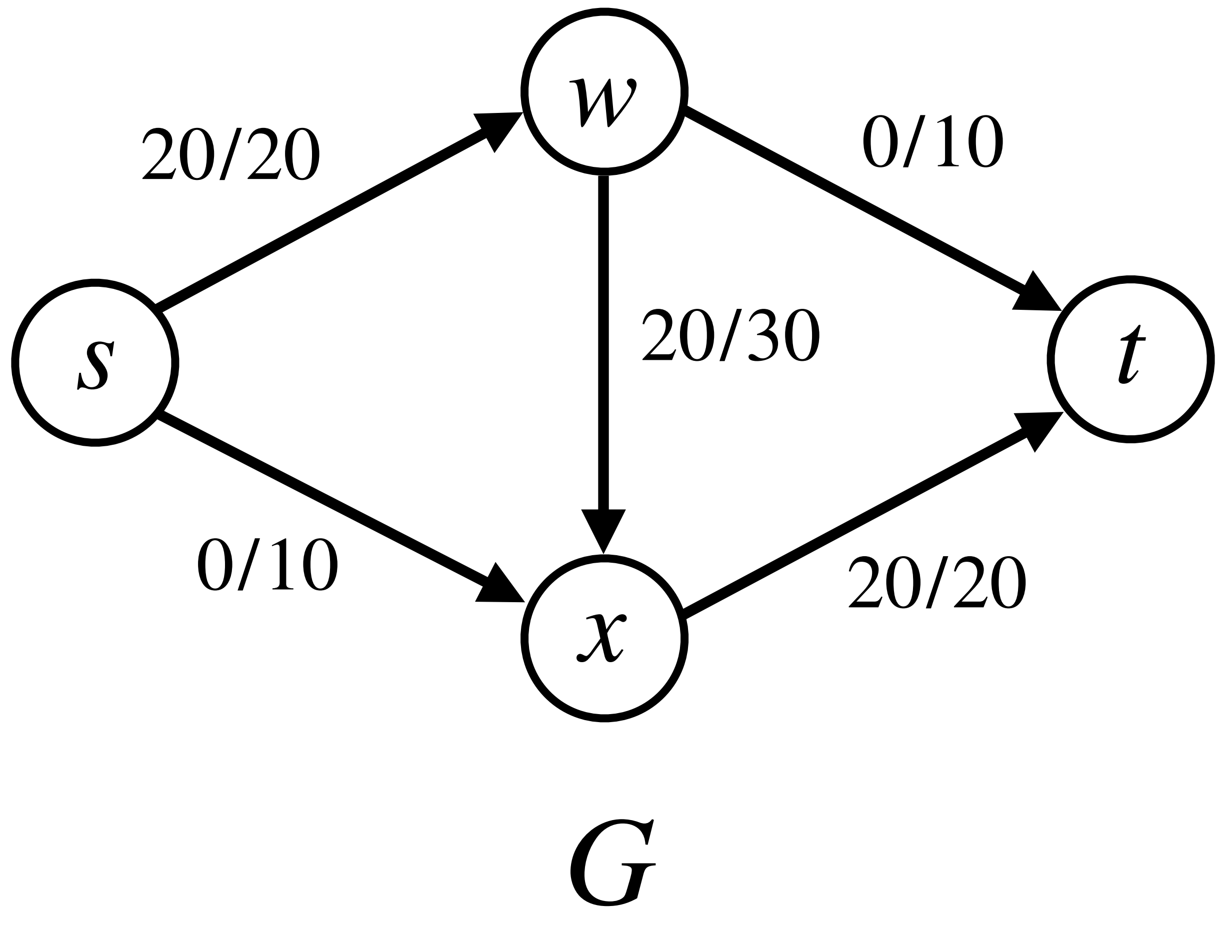
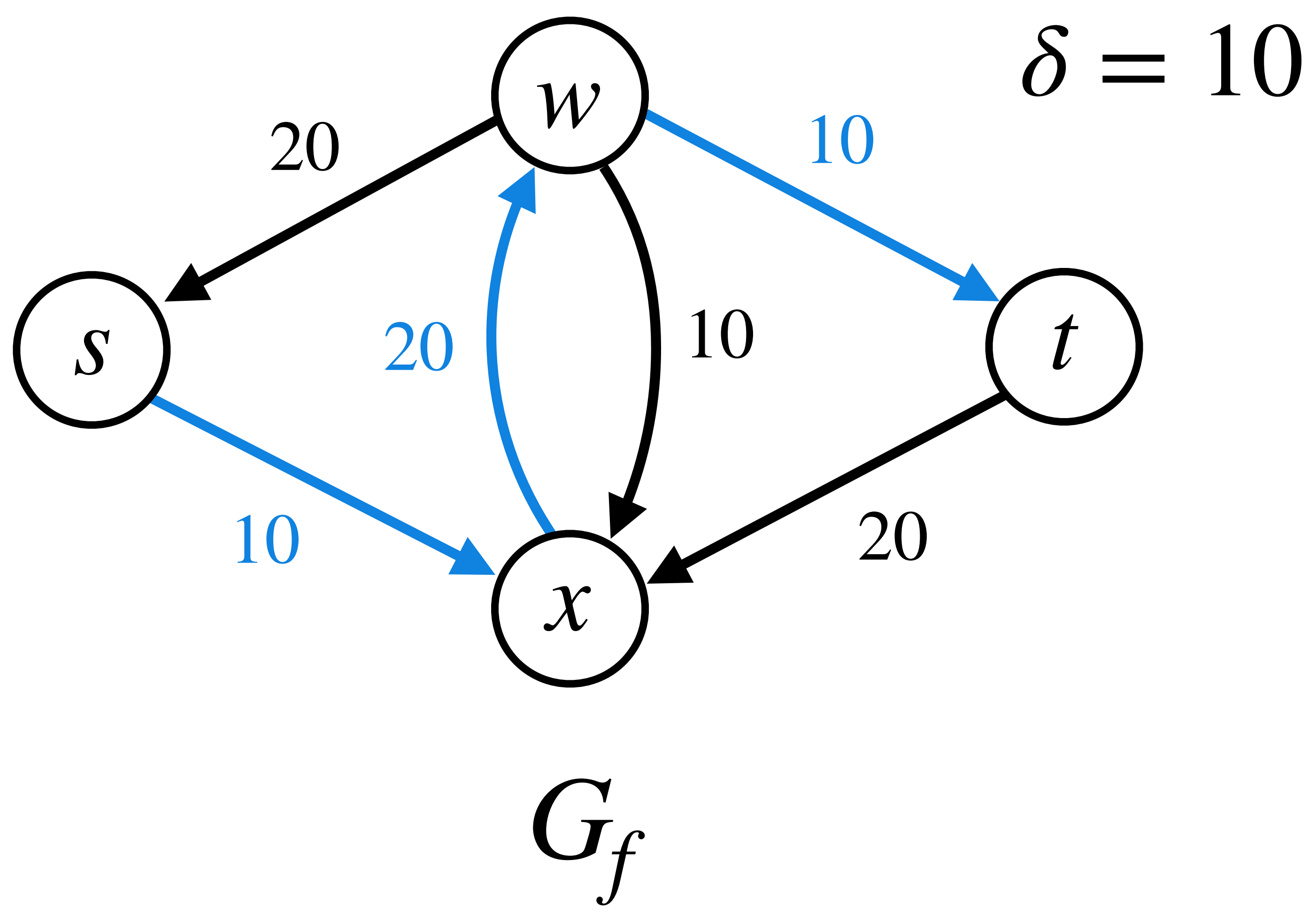
Example:





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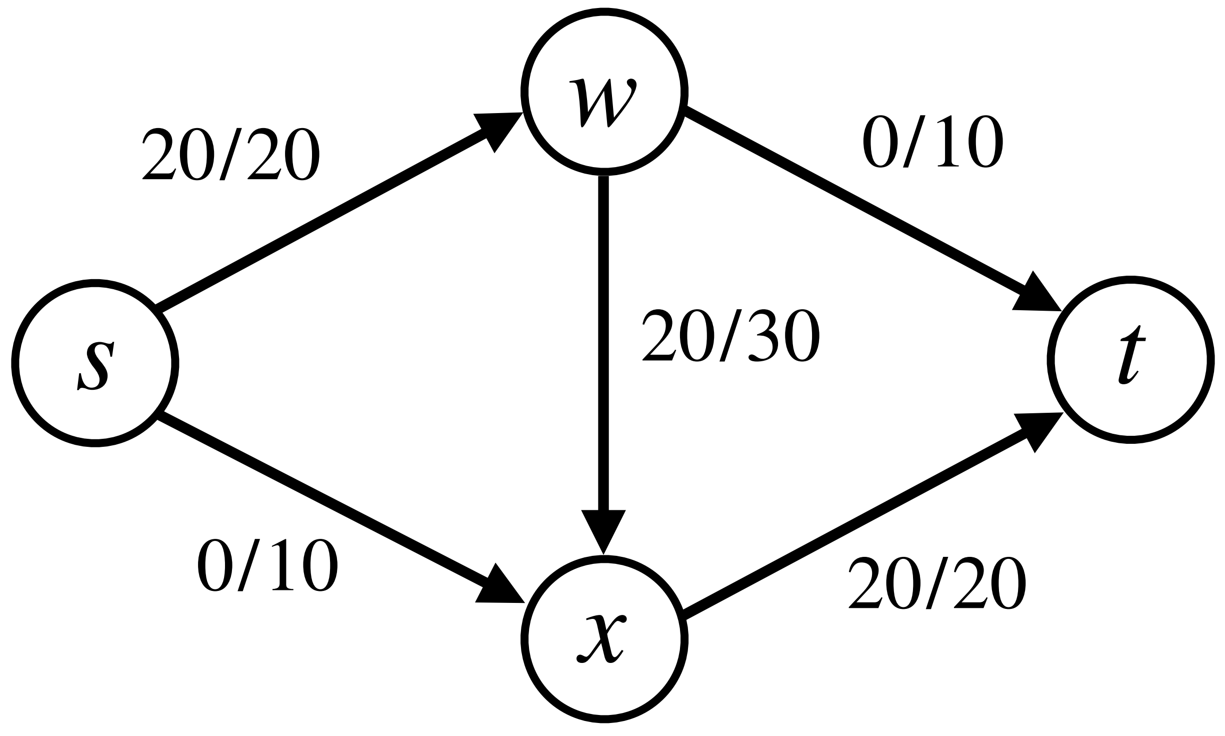

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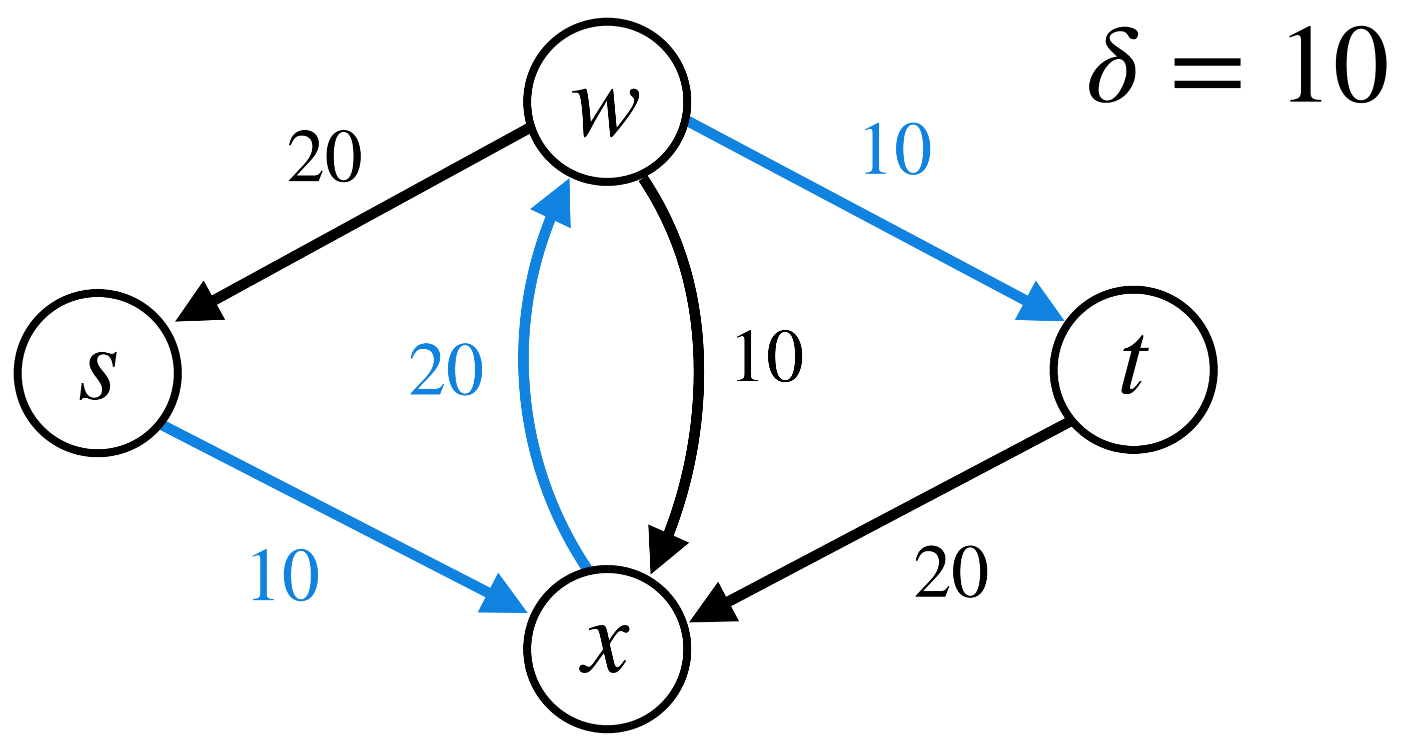

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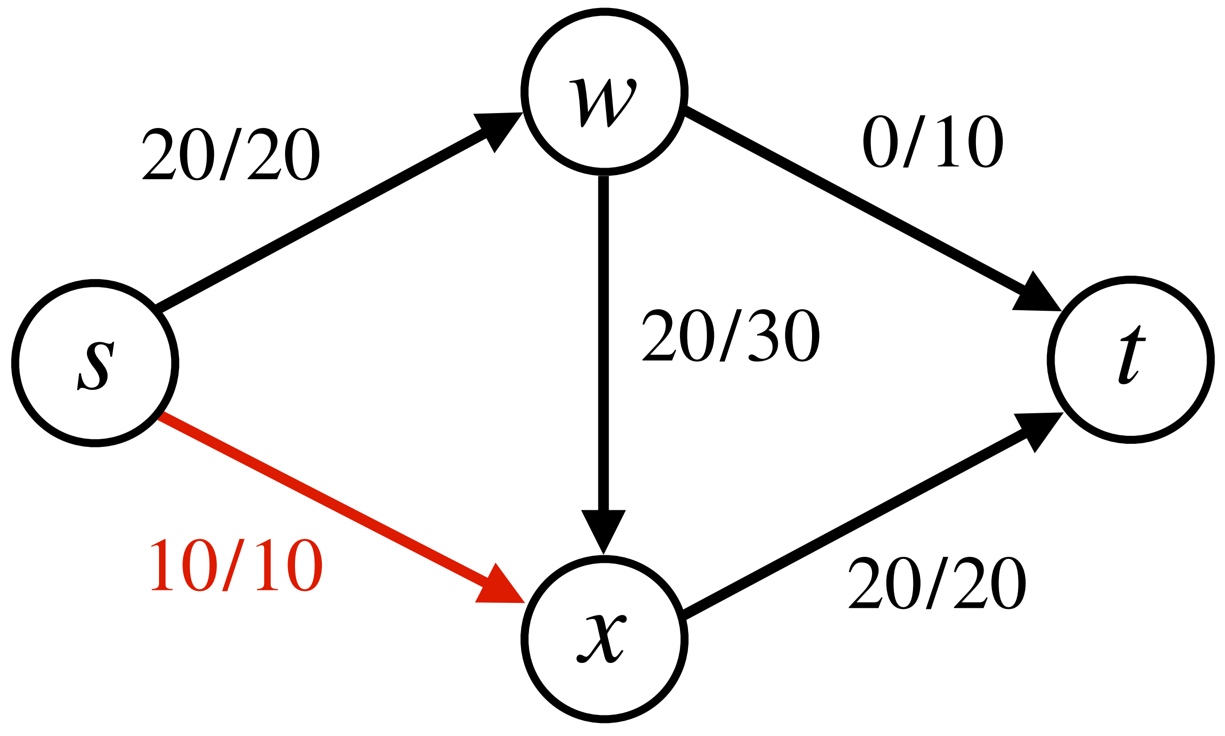

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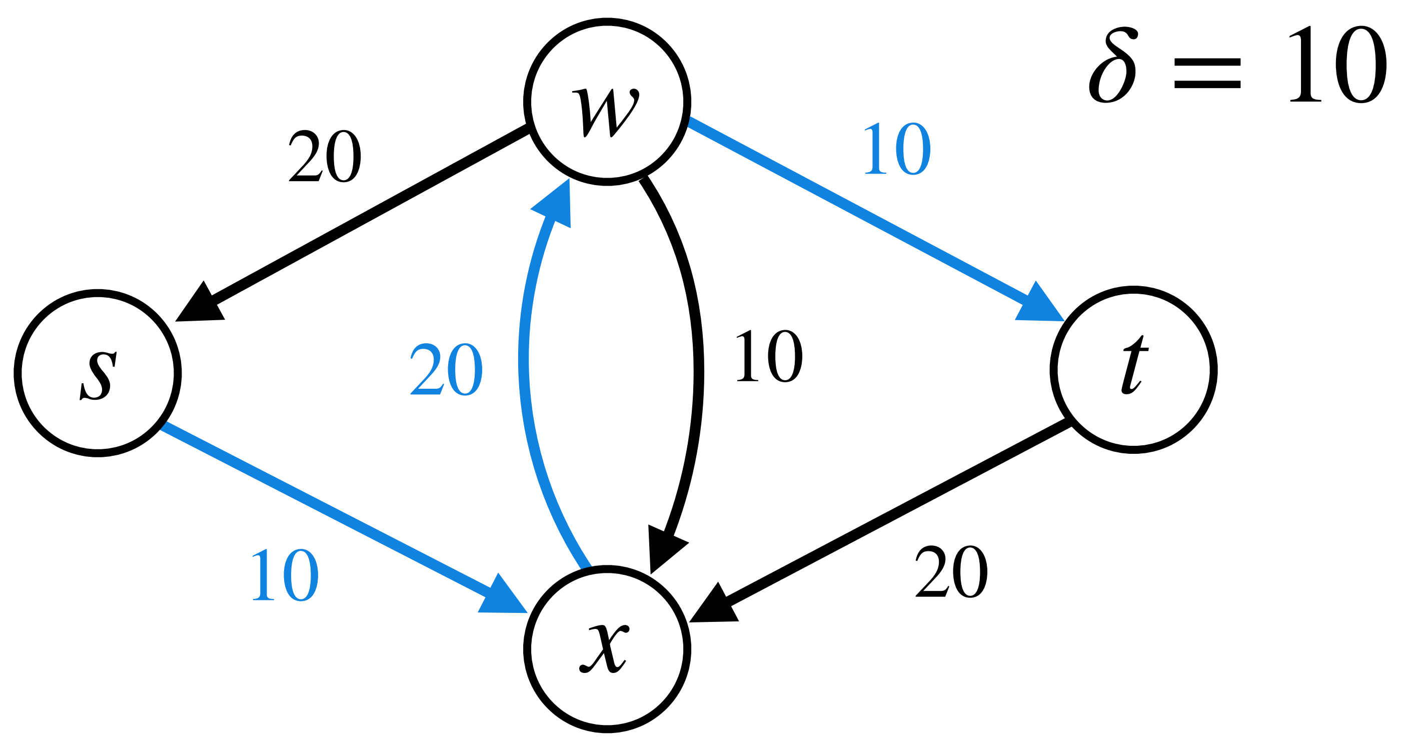

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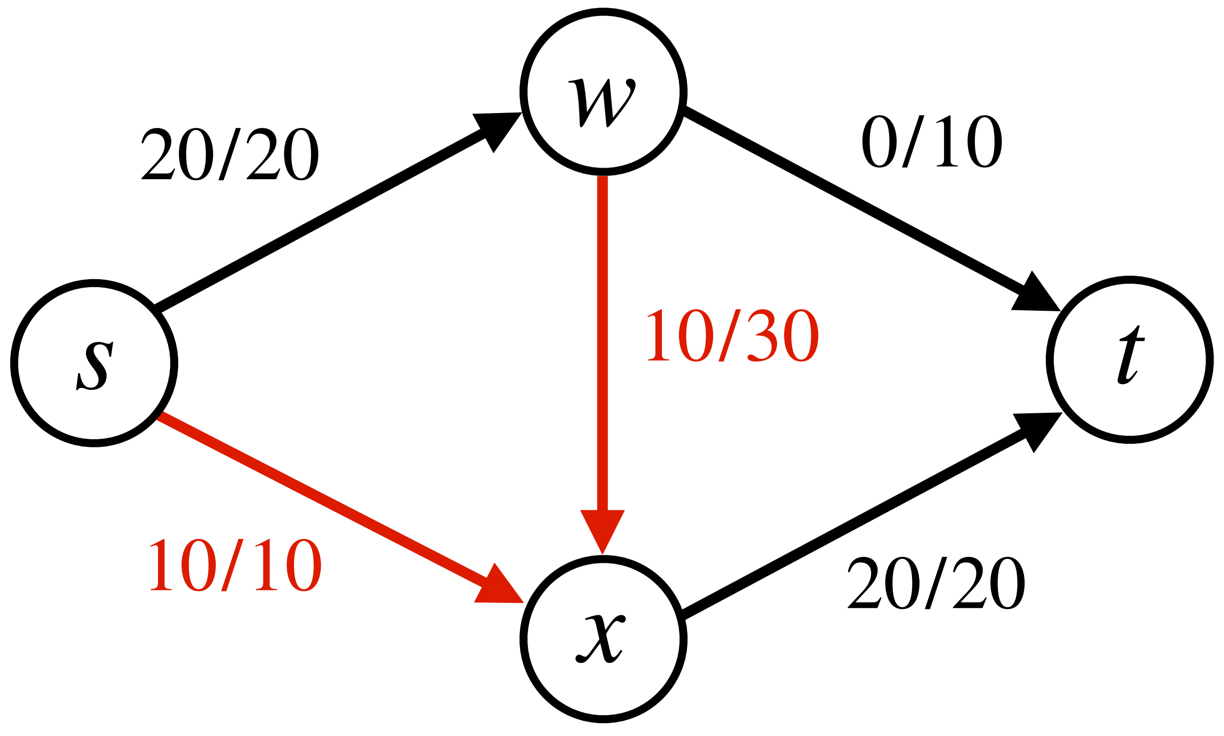

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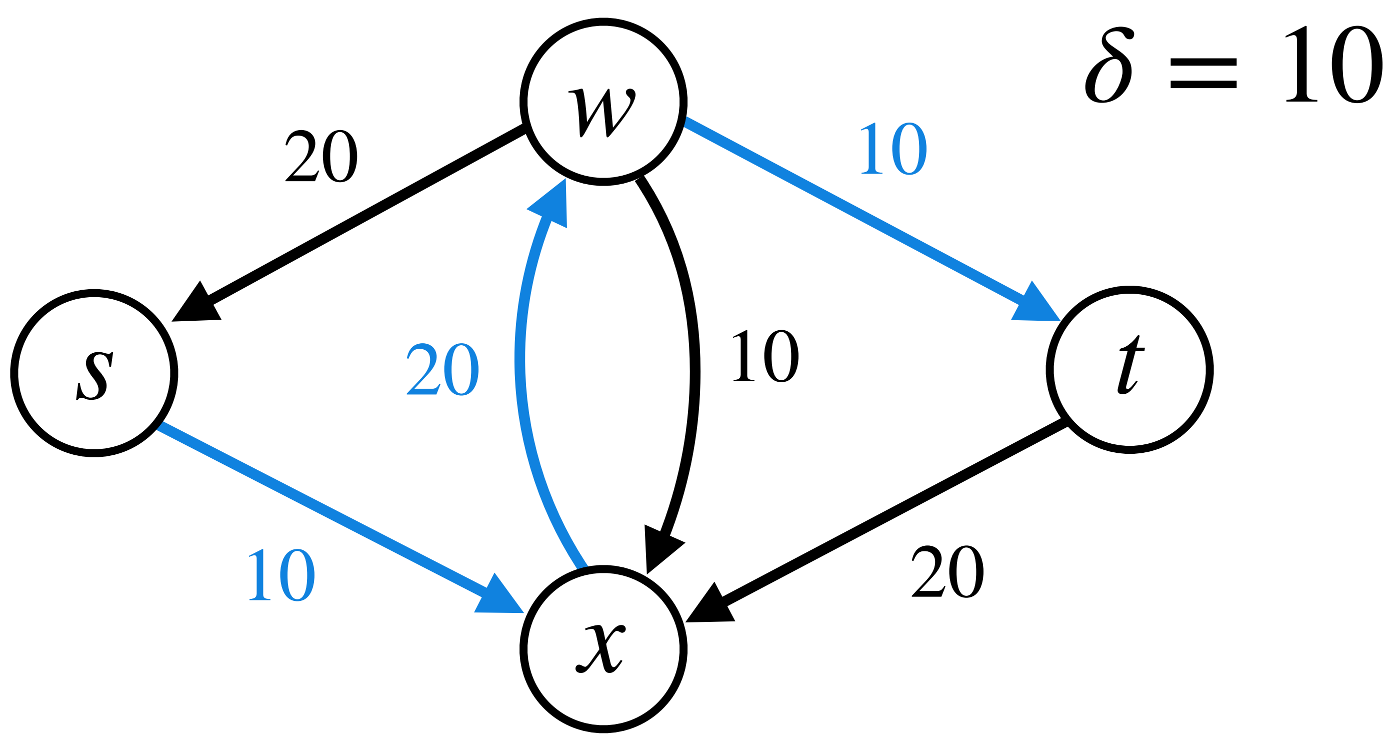

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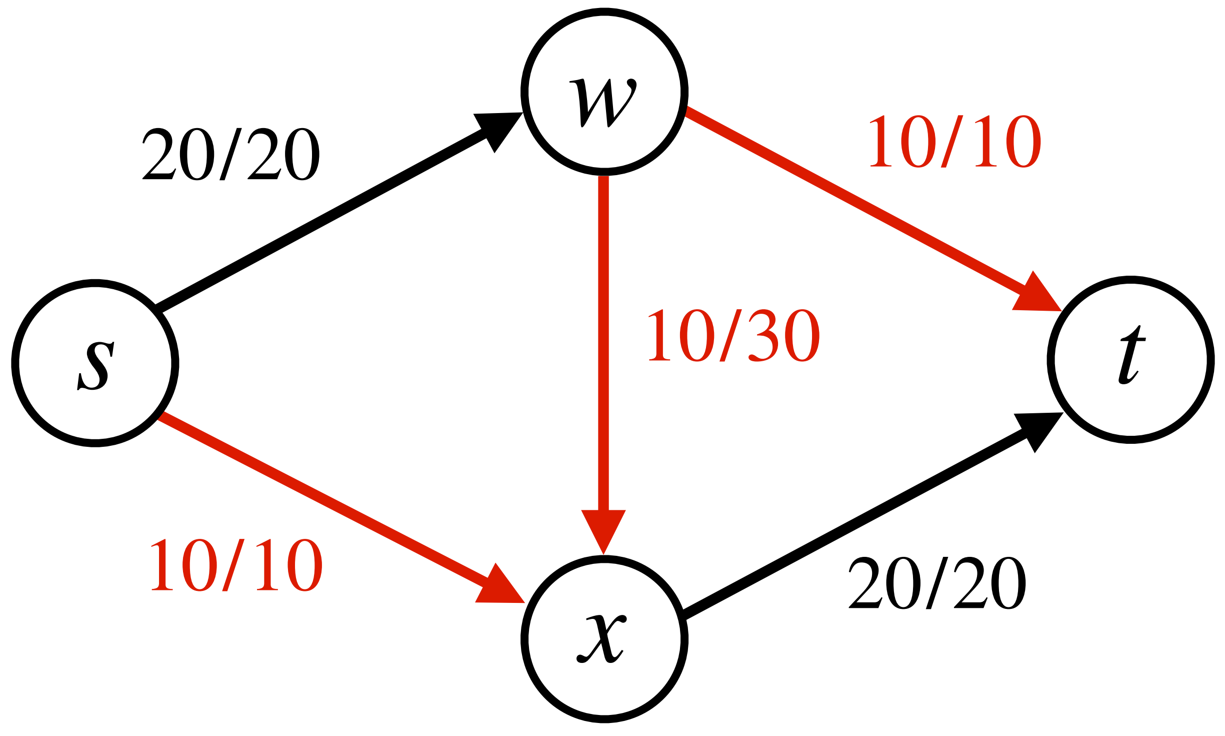

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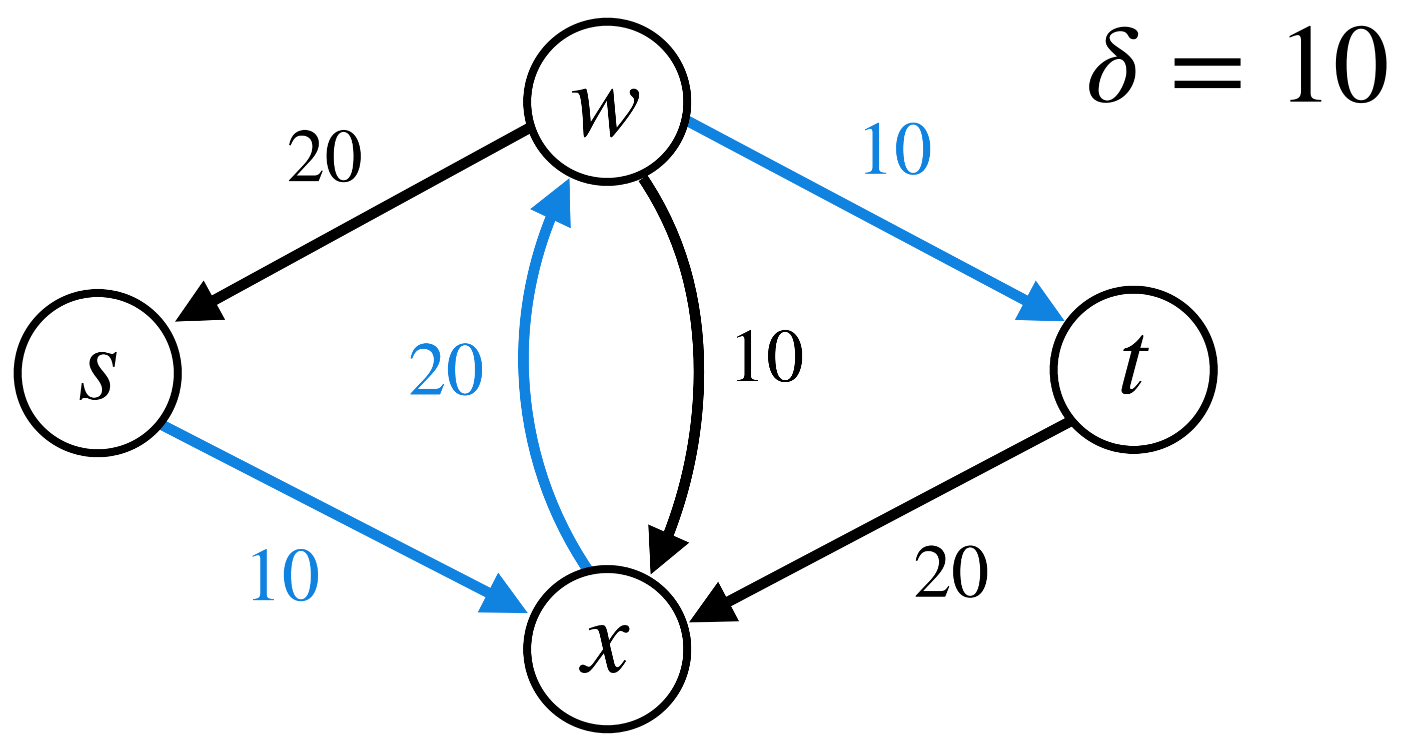

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What about capacity, conservation constraints?

Example: